The sharp for the Chang model is small

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This paper is dedicated to the memory of Rich Laver and Jim Baumgartner, who I treasured as friends, colleagues and exemplars since we were graduate students together.

Abstract

Woodin has shown that if there is a measurable Woodin cardinal then there is, in an appropriate sense, a sharp for the Chang model. We produce, in a weaker sense, a sharp for the Chang model using only the existence of a cardinal κ having an extender of length $\kappa^{+\omega_1}$.

Contents

1 Introduction

The Chang model, introduced in [1], is the smallest model of ZF set theory which contains all countable sequences of ordinals. It may be constructed as $L(^\omega\Omega)$, that is, by imitating the recursive definition of the L_α hierarchy, setting $\mathbb{C}_0 = \varnothing$ and $\mathbb{C}_{\alpha+1} = \mathrm{Def}^{\mathbb{C}_\alpha}(\mathbb{C}_\alpha)$, but modifying the definition for limit ordinals α by setting $\mathbb{C}_\alpha = [\alpha]^{<\omega_1} \cup \bigcup_{\alpha' < \alpha} \mathbb{C}_\alpha$. Alternatively it may be constructed, as did Chang, by replacing the use of first order logic in the definition of L with the infinitary logic L_{ω_1,ω_1}. We write $\mathbb{C}$ for the Chang model.

Clearly the Chang model contains the set $\mathcal{R}$ of reals, and hence is an extension of $L(\mathcal{R})$. Kunen [7] has shown that the axiom of choice fails in the Chang model whenever there are uncountably many measurable cardinals; in particular the theory of $\mathbb{C}$ may vary, even when the set of reals is held fixed. We show that in the presence of sufficiently large cardinal strength this is not true. An earlier unpublished result of Woodin states that if there is a Woodin limit of Woodin cardinals, then there is a sharp for the Chang model. Our result is not strictly comparable to Woodin's, since although ours uses a much smaller cardinal, Woodin's notion of a sharp is stronger, and his result gives the sharp for a stronger model. Perhaps the most striking aspect of the new result is its characterization of the size of the Chang model. Although the Chang model, like $L(\mathcal{R})$, can have arbitrary large cardinal strength coded into the reals, the large cardinal strength of $\mathbb{C}$ relative to $L(\mathcal{R})$, even in the presence of large cardinals in V, is at most $o(\kappa) = \kappa^{+\omega_1} + 1$.

The next three definitions describe our notion of a sharp for $\mathbb{C}$. Following this definition and a formal statement of our theorem, we will more specifically discuss the differences between our result and that of Woodin.

As with traditional sharps, the sharp for the Chang model asserts the existence of a closed, unbounded class I of indiscernibles. The conditions on I are given in Definition 1.3, following two preliminary definitions:

Definition 1.1. Say that a subset B of a closed class I is *suitable* if (a) B is countable and closed, (b) every member of B which is a limit point of I

of countable cofinality is also a limit point of B, and (c) B is closed under immediate predecessors in I.

We say that suitable sequences B and B' are *equivalent* if they have the same order type and, writing $\sigma \colon B \to B'$ for the order isomorphism, $\forall \kappa \in B\ \sigma(\kappa) \in \lim(I) \iff \kappa \in \lim(I)$.

Note that if B is suitable and β' is the successor of β in B, then either β' is the successor of β in I, or else β' is a limit member of I and $\mathrm{cf}(\beta') > \omega$. Indeed clauses (b) and (c) of the definition of a suitable sequence are equivalent to the assertion that every gap in B, as a subset of I, is capped by a member of B which is a limit point of I of uncountable cofinality.

Definition 1.2. Suppose that T is a collection of constants and functions with domain in $[\kappa]^n$ for some $n < \omega$. Write $\mathcal{L}_T$ for the language of set theory augmented with symbols denoting the members of T. A *restricted formula* in the language $\mathcal{L}_T$ is a formula φ such that every variable occurring inside an argument of a function in T is free in φ.

Definition 1.3. We say that there is a *sharp for the Chang model* $\mathbb{C}$ if there is a closed unbounded class I of ordinals and a set T of functions having the following three properties:

1. Suppose that B and B' are equivalent suitable sets, and let $\varphi(B)$ be a restricted formula. Then

$$\mathbb{C} \models \varphi(B) \iff \varphi(B').$$

2. Every member of $\mathbb{C}$ is of the form $\tau(B)$ for some term $\tau \in T$ and some suitable sequence B.

3. If V' is any universe of ZF set theory such that $V' \supseteq V$ and $\mathcal{R}^{V'} = \mathcal{R}^V$ then, for all restricted formulas φ

$$\mathbb{C}^{V'} \models \varphi(B) \iff \mathbb{C}^V \models \varphi(B).$$

for any $B \subseteq I$ which is suitable in both V and V'.

Note, in clause 3, that $\mathbb{C}^{V'}$ may be larger than $\mathbb{C}^V$. A sequence B which is suitable in V may not be suitable in V', as a limit member of B may have uncountable cofinality in V but countable cofinality in V'. However the class I, as well as the theory, will be the same in the two models.

The sharp defined here is somewhat provisional, as is suggested by the gap between the upper and lower bounds in Theorem 1.5. The major consequence of $0^\sharp$ which is shared by this notion of a sharp is the existence of nontrivial embeddings of $\mathbb{C}$:

Proposition 1.4. *Suppose that I is a class satisfying Definition 1.3 and $\sigma \colon I \to I$ is an increasing map which* (i) *is continuous at limit points of cofinality ω, and for all $\kappa \in B$* (ii) $\sigma(\min(I\backslash(\kappa+1))) = \min(I\backslash(\sigma(\kappa)+1))$ *and* (iii) $\sigma(\kappa)$ *is a limit point of I if and only if κ is a limit point of I. Then σ can be extended to an elementary embedding $\sigma^* \colon \mathbb{C} \to \mathbb{C}$.* $\qquad\square$

Definition 1.3 is not strong enough to imply the converse, that any elementary embedding $\sigma^*\colon \mathbb{C} \to \mathbb{C}$ is generated by some such map $\sigma\colon I \to I$, and it does not imply that the embeddings σ^* are unique. Note, for example, that if a sharp for $\mathbb{C}$ is given, according to Definition 1.3, by I and T then $I' = \{\, \kappa_{\nu\cdot\omega_1} \mid \nu \in \Omega \,\}$ also satisfies the definition, using the set $T' = T \cup \{\, t_\alpha \mid \alpha < \omega_1 \,\}$ of terms where $t_\alpha(\kappa_{\omega_1\cdot\nu}) = \kappa_{\omega_1\cdot\nu+\alpha}$. However, the restriction to I' of the embedding $i^*\colon \mathbb{C} \to \mathbb{C}$ induced by the embedding $i\colon \kappa_{\omega_1\cdot\nu+\alpha} \mapsto \kappa_{\omega_1\cdot(\nu+1)+\alpha}$ does not satisfy the hypothesis of Proposition 1.4. It is likely that this deficiency will eventually be resolved by a characterization of the "minimal sharp", that is, of the weakest large cardinal (or the smallest mouse) which yields a sharp in the sense of Definition 1.3.

Recall that a traditional sharp, such as $0^\sharp$, may be viewed in either of two different ways: as a closed and unbounded class of indiscernibles which generates the full (class) model, or as a mouse with a final extender on its sequence which is an ultrafilter.

From the first viewpoint, perhaps the most striking difference between $0^\sharp$ and our sharp for $\mathbb{C}$ is the need for external terms in order to generate $\mathbb{C}$ from the indiscernibles. From the second viewpoint, regarding the sharp as a mouse, the sharp for the Chang model involves two modifications:

1. For the purposes of this paper, a *mouse* will always be a mouse over the reals, that is, an extender model of the form $J_\alpha(\mathcal{R})[\mathcal{E}]$.

2. The final extender of the mouse which represents the sharp of the Chang model will be a proper extender, not an ultrafilter.

It is still unknown how large the final extender must be. We show that its length is somewhere in the range from $\kappa^{+(\omega+1)}$ to $\kappa^{+\omega_1}$, inclusive:

Theorem 1.5 (Main Theorem). *1. Suppose that there is no mouse $M = J_\alpha(\mathcal{R})[\mathcal{E}]$ with a final extender $E = \mathcal{E}_\gamma$ with critical point κ and length $\kappa^{+(\omega+1)}$ in $J_\alpha(\mathcal{R})[\mathcal{E}]$ such that $\mathrm{cf}^V(\mathrm{length}(E)) > \omega$. Then $K(\mathcal{R})^{\mathbb{C}}$, the core model over the reals as defined in the Chang model, is an iterated ultrapower (without drops) of $K(\mathcal{R})^V$; and hence there is no sharp for the Chang model.*

2. Suppose that there is a model $L(\mathcal{R})[\mathcal{E}]$ which contains all of the reals and has an extender E of length $(\kappa^{+\omega_1})^{L(\mathcal{R})[\mathcal{E}]}$, where κ is the critical point of E. Then there is a sharp for $\mathbb{C}$.

This problem was suggested by Woodin in a conversation at the Mittag-Lefler Institute in 2009, in which he observed that there was an immense gap between the hypothesis needed for his sharp, and easily obtained lower bounds such as a model with a single measure. At the time I conjectured that the same argument might show that any extender model would provide a similar lower bound, but James Cummings and Ralf Schindler, in the same conversation, pointed out that Gitik's results suggest that it would fail at an extender of length $\kappa^{+(\omega+1)}$.

I would also like to thank Moti Gitik, for suggesting his forcing for the proof of clause 2 and explaining its use. I have generalized his forcing to add new sequences of arbitrary countable length. I have also made substantial but, I believe, inessential changes to the presentation; I hope that he will recognize his forcing in my presentation. Many of the arguments in this paper, indeed almost all of those which do not directly involve either the generalization of the forcing or the application to the Chang model, are due to Gitik.

1.1 Comparison with Woodin's sharp

Our notion of a sharp for $\mathbb{C}$ differs from that of Woodin in several ways. We will discuss them in roughly increasing order of importance.

1. The theory of our sharp can depend on the set of reals, while the theory of Woodin's sharp does not; however this is due to the large cardinals involved, rather than the definition of the sharp. Woodin's proof that the theory of $L(\mathcal{R})$ is invariant under set forcing also shows that the theory of our sharp stabilizes in the presence of a class of Woodin cardinals.

Two differences which might seem to be weaknesses in our model are actually only differences in presentation.

2. Woodin's indiscernibles are defined to be indiscernible in the infinitary language L_{ω_1,ω_1}, whereas we use only first order logic. However the two languages are equivalent in this context: since $\mathbb{C}$ is closed under countable sequences and $\mathbb{C}_\alpha \prec \mathbb{C}$ whenever α is a member of the class I of indiscernibles, the existence of our sharp implies that any formula of L_{ω_1,ω_1} is equivalent to a formula of first order logic having a parameter which is a countable sequence of ordinals.

3. For Woodin's sharp, any two subsequences of I are indiscernible, while for our sharp only "suitable" subsequences are considered. The requirement of suitability could be eliminated by replacing I with the class of limit points of I of uncountable cofinality, and making a corresponding addition to the class T of terms, but it seems that doing so would ultimately lose information about the structure of the sharp. This point is discussed further in Subsection 3.1.

The final two differences are significant. The first can probably be removed, while the second is basic and explains the difference in the hypotheses used:

4. The notion of restricted formulas is entirely absent from Woodin's results: he allows the terms from T to be used as full elements of the language. We believe that our need for restricted formulas is due to the choice of terms and will eventually be removed by a more complete analysis resolving the question about the size of the minimal mouse needed to give a sharp for $\mathbb{C}$. If this conjecture turns out to be incorrect then its failure would be a major weakness in our notion of a sharp.

5. Woodin has observed, in a personal communication, that his sharp actually is a sharp for a much stronger model, namely the smallest model which contains all countable sequences of ordinals and the stationary filter on

the set $\mathcal{P}_{\omega_1}([\lambda]^\omega)$ for every λ. Thus our constructions do not conflict, but instead describe sharps for different models, and this explains the difference in the hypotheses needed.

Woodin has observed (private communication) that some of the gap between the two sharps can be filled by modifying the construction of this paper to use the least mouse M over the reals such that M has infinitely many Woodin cardinals below the extenders needed for the conclusion of Theorem 1.5(2). This would give a version of our sharp which can be coded by a set $X \subseteq \mathcal{R}$ having the following property: Suppose that V' is any inner model of V such that $X \cap V' \in V'$. Then $X \cap V'$ codes the corresponding sharp for the Chang model of V'. Woodin regards this as the "true sharp"; however it seems that the better terminology would be tof regard this not as the analog of the sharp operator, but as the analog of the M_ω mouse operator.

Future work, and the publication of Woodin's work on his sharp, will be needed to better comprehend the possibilities of extensions of sharps for Chang-like models in analogy with the extended theory related to $0^\sharp$. At the same time, as points 3 and 4 above make clear, further work is needed towards clarifying the basic notion of the sharp for the Chang model as presented in this paper.

1.2   Some basic facts about $\mathbb{C}$

As pointed out earlier, the Axiom of Choice fails in $\mathbb{C}$ if there are infinitely many measurable cardinals. However, the fact that $\mathbb{C}$ is closed under countable sequences implies that the axiom of Dependent Choice holds, and this is enough to avoid most of the serious pathologies which can occur in a model without choice. For life without Dependent Choice, see for example [5], which gives a model with surjective maps from $\mathcal{P}(\aleph_\omega)$ onto an arbitrarily large cardinal λ without any need for large cardinals.

The same argument that shows that every member of L is ordinal definable implies that every member of $\mathbb{C}$ is definable in $\mathbb{C}$ using a countable sequence of ordinals as parameters.

In the proof of part 1 of Theorem 1.5 we make use of the core model $K(\mathcal{R})$ inside of $\mathbb{C}$, and in the absence of the Axiom of Choice this requires some justification. In large part the Axiom of Choice can be avoided in the construction and theory of this core model, since the core model itself is well ordered (after using countably complete forcing to map the reals onto ω). However one application of the Axiom of Choice falls outside of this situation: the use of Fodor's pressing down lemma, the proof of which requires choosing closed unbounded sets as witnesses that the sets where the function is constant are all nonstationary. This lemma is needed in the construction of $K(\mathcal{R})$ in order to prove that the comparison of pairs of mice by iterated ultrapowers always terminates. However, this is not a problem in the construction of $K(\mathcal{R})$ in $\mathbb{C}$, as we can apply Fodor's lemma in the universe V, which satisfies the Axiom of Choice, to verify that all comparisons terminate.

The proof of the covering lemma involves other uses of Fodor's lemma; however we do not use the covering lemma.

1.3 Notation

We use generally standard set theoretic notation. We use Ω to mean the class of all ordinals, and frequently treat Ω itself as an ordinal. If h is a function, then we use $h[B]$ for the range of h on B, $h[B] = \{ h(b) \mid b \in B \}$. We write $[X]^{\kappa}$ for the set of subsets of X of size κ.

In forcing, we use $p < q$ to mean that p is stronger than q. The notation $p \parallel \varphi$ means that the condition p decides φ, that is, either $p \Vdash \varphi$ for $p \Vdash \neg\varphi$. If P is a forcing order and $s \in P$, then we write $P\|_s$ for the forcing below s, that is, the restriction of P to $\{ t \in P \mid t \leqslant s \}$.

If E is an extender, then we write $\mathrm{supp}(E)$ for the support, or set of generators, of E. Typically we take this to be the interval $[\kappa, \mathrm{length}(E))$ where κ is the critical point of E; however we frequently make use of the restriction of E to a nontransitive[1] set of generators: that is, if $S \subseteq \mathrm{supp}(E)$ then we write $E \Cap S$ for the restriction of E to S, so $\mathrm{Ult}(V, E\Cap S) \cong \{ i^E(f)(a) \mid f \in V \wedge a \in [S]^{<\omega} \}$. We remark that $\mathrm{Ult}(V, E\Cap S) = \mathrm{Ult}(V, \bar{E})$, where $\bar{E}$ is the *transitive collapse of $E\Cap S$*, that is, the extender obtained from $E\Cap S$ by using the transitive collapse $\sigma \colon [\kappa, \mathrm{length}(\bar{E})) \cong \mathrm{supp}(E) \cap \{ i^E(f)(a) \mid a \in [S]^{<\omega} \}$ and setting the ultrafilter $(\bar{E})_\alpha = E_{\sigma^{-1}(\alpha)}$. In cases where the $E\Cap S \notin M$ but the transitive collapse $\bar{E} \in M$, we frequently describe constructions as using $E\Cap S$ when the actual construction inside M must use $\bar{E}$. Such use will not always be explicitly stated.

We write $(E)_a$ for the ultrafilter $\{ x \subseteq H_{\mathrm{crit}(E)} \mid a \in i^E(x) \}$.

We make extensive use of the core model over the reals, $K(\mathcal{R})$. However we make no (direct) use of fine structure, largely because we make no attempt to use the weakest hypothesis which could be treated by our argument. The reader will need to be familiar with extender models, but only those weaker than strong cardinal, that is, without the complications of overlapping extenders and iteration trees. For our purposes, a mouse will be an extender model $M = J_\alpha(\mathcal{R})[\mathcal{E}]$, where $\mathcal{R}$ is the set $\mathcal{P}(\omega)$ of reals and $\mathcal{E}$ is a sequence of extenders, and it generally can be assumed to be a model of Zermelo set theory (and therefore equal to $L_\alpha(\mathcal{R})[\mathcal{E}]$).

The ultrafilters in a mouse M over the reals, including those appearing as components of an extender, are all complete over sets of reals. That is, if U is an ultrafilter and $f \colon X \to \mathcal{P}(\mathcal{R})$ for some $X \in U$ then there is a set $a \subseteq \mathcal{R}$ such that $\{ x \in X \mid f(x) = a \} \in U$. This implies the needed instances of the Axiom of Choice:

Proposition 1.6. *Suppose that U is an ultrafilter and $X \in U$. Then*

1. *there is a well orderable $X' \subseteq X$ such that $X' \in U$, and*

2. *if f is a function such that $\{ x \in X \mid f(x) \neq \varnothing \} \in U$ then there is a function g such that $\{ x \in X \mid g(x) \in f(x) \} \in U$.*

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[1] We regard $\mathrm{supp}(E) = [\kappa, \lambda)$ as "transitive" despite its omission of ordinals less than κ. We could equivalently, but slightly less conveniently, use $\mathrm{supp}(E) = \mathrm{length}(E)$.

Proof. Every element of M is ordinal definable from a real parameter. If $x \in M$, then let φ_x be the least formula φ, with ordinal parameters, such that $(\exists r \in \mathcal{R})\forall z \ (\varphi(z,r) \iff z = x)$, and let $R_x = \{ r \in \mathcal{R} \mid \forall z \ (\varphi_x(z,r) \iff z = x) \}$. For the first clause, there is $R \subseteq \mathcal{R}$ such that $X' = \{ x \in X \mid R_x = R \} \in U$. Thus, if r is any member of R, then every member of X' is ordinal definable from r.

The proof of the second clause is similar, using $R \subseteq \mathcal{R}$ such that $\{ x \in X \mid \bigcup_{z \in f(x)} R_z = R \}$. $\qquad\square$

If $M = J_\alpha(\mathcal{R})[\mathcal{E}]$ is a mouse then we write $M|\gamma$ for $J_\gamma(\mathcal{R})[\mathcal{E}\restriction\gamma]$, that is, for the cut off of M at γ without including the active final extender $\mathcal{E}_\gamma$ if there is one. This is most commonly used as $N|\Omega$, where N is the final model of an iteration of length Ω and $\Omega^N > \Omega$.

2   The Lower bound

The proof of Theorem 1.5(1), giving a lower bound to the large cardinal strength of a sharp for the Chang model, is a straightforward application of a technique of Gitik (see the proof of Lemma 2.5 for $\delta = \omega$ in [6]).

Proof of Theorem 1.5(1). The proof of the lower bound uses iterated ultrapowers to compare $K(\mathcal{R})$ with $K(\mathcal{R})^{\mathbb{C}}$. Standard methods show that $K(\mathcal{R})^{\mathbb{C}}$ is not moved in this comparison, so there is an iterated ultrapower $\langle M_\nu \mid \nu \leqslant \theta \rangle$, For some $\theta \leqslant \Omega$, such that $M_0 = K(\mathcal{R})$ and $M_\theta = K(\mathcal{R})^{\mathbb{C}}$. This iterated ultrapower is defined by setting (i) $M_\alpha = \mathrm{dir}\lim\{ M_{\alpha'} \mid \alpha'' < \alpha' < \alpha \}$ for sufficiently large $\alpha'' < \alpha$ if α is a limit ordinal, and (ii) $M_{\alpha+1} = \mathrm{Ult}(M_\alpha^*, E_\alpha)$, where E_α is the least extender in M_α which is not in $K(\mathcal{R})^{\mathbb{C}}$ and M^* is equal to M_α unless E_α is not a full extender in M_α, in which case M_α^* is the largest initial segment of M_α in which E_α is a full extender.

We want to show that (i) this does not drop, that is, $M_\alpha^* = M_\alpha$ for all α, and (ii) $M_\theta = K(\mathcal{R})^{\mathbb{C}}$.

If either of these is false, then $\theta = \Omega$ and there is a closed unbounded class C of ordinals α such that $\mathrm{crit}(E_\alpha) = \alpha = i_\alpha(\alpha)$. Since $o(\kappa) < \Omega$ for all κ it follows that there is a stationary class $S \subseteq C$ of ordinals of cofinality ω such that $i_{\alpha',\alpha}(E_{\alpha'}) = E_\alpha$ for all $\alpha' < \alpha$ in S. Fix $\alpha \in S \cap \lim(S)$; we will show that the hypothesis of Theorem 1.5(1) implies that $E_\alpha \in \mathbb{C}$, contradicting the choice of E_α.

To this end, let $\vec{\alpha} = \langle \alpha_n \mid n \in \omega \rangle$ be an increasing sequence of ordinals in S such that $\bigcup_{n \in \omega} \alpha_n = \alpha$. We call a sequence $\langle \beta_n \mid n \in \omega \rangle$ a *thread* for the generator β of E_α if $\beta_n = i_{\alpha_n,\alpha}^{-1}(\beta)$ for all sufficiently large $n < \omega$. The technique of Gitik used in [6, Lemma 2.3] gives a formula φ such that $\varphi(\vec{\alpha}, \vec{\beta}, \beta)$ holds if and only if $\beta < \kappa^{+\omega}$ and $\vec{\beta}$ is a thread for β. Since all of the threads are in $\mathbb{C}$, this implies that $E_\alpha \restriction \kappa^{+\omega} \in \mathbb{C}$. If $\gamma = \mathrm{length}(E_\alpha) < \kappa_\alpha^{+(\omega+1)}$ then this construction can be extended to all of E_α by using $\langle i_{\alpha_n}^{-1}(\mathrm{length}(E_\alpha)) \mid n \in \omega \rangle$

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as an additional parameter. But the hypothesis of Theorem 1.5(1) implies that length$(E_\alpha) < (\kappa_\alpha^{+\omega})^\mathbb{C}$, so $E_\alpha \in \mathbb{C}$, contradicting the definition of E_α.

It follows that no sharp for $\mathbb{C}$ exists, as otherwise the embedding given by Proposition 1.4 would make an iterated ultrapower of $K(\mathcal{R})$ non-rigid. □

3 The upper bound

The proof of Theorem 1.5(2) will take up the rest of this paper except for the final Section 5, which poses some open questions.

The hypothesis of Theorem 1.5(2) is stronger than necessary: our construction of the sharp for $\mathbb{C}$ uses only a sufficiently strong mouse over the reals, that is, a model $M = J_\gamma(\mathcal{R})[\mathcal{E}]$ where $\mathcal{E}$ is an iterable extender sequence.

At this point we describe a general procedure for constructing a sharp from a mouse. For this purpose we will assume that M is a mouse satisfying the following conditions: (i) $|M| = |\mathcal{R}|$, definably over M, indeed (ii) there is an onto function $h \colon \mathcal{R} \to M$ which is the union of an increasing ω_1 sequence of functions in M, and (iii) M has a last $(\kappa, \kappa^{+\omega_1})$-extender, $E \in M$. We can easily find such a mouse from the hypothesis of Theorem 1.5(2) by choosing a model N of the form $J_\gamma(\mathcal{R})[\mathcal{E}]$ with the last two properties and letting M be the transitive collapse of the Skolem hull of $\mathcal{R} \cup \omega_1$ in N. In Definition 4.1, at the start of section 4, we will make additional and more precise assumptions on M which are used in the proof of the Main Theorem.

We remark that we could assume the Continuum Hypothesis by generically adding a map g mapping ω_1 onto the reals. Doing so would not add any new countable sequences and hence would not affect the Chang model. Indeed we could use $J_\gamma[g][\mathcal{E}]$ for the mouse M instead of $J_\gamma(\mathcal{R})[\mathcal{E}]$, so that M satisfies the Axiom of Choice and the Continuum Hypothesis, along with all of the properties we require of M. We do not do so (though we will need to generically add such a map g near the end of the proof) but the reader certainly may, if desired, assume that this has been done.

The following simple observation is basic to the construction:

Proposition 3.1. *The mouse M is closed under countable subsequences.*

Proof. By the assumption (b) on M, any countable subset $B \subseteq M$ is equal to $h[b]$ for a function $h \in M$ and set $b \subset \mathcal{R}$. Since M contains all reals, and any countable set of reals can be coded by a single real, $b \in M$ and thus $B \in M$. □

As in the case of $0^\sharp$, we obtain the sharp for the Chang model by iterating the final extender E out of the universe:

Definition 3.2. We write $i_\alpha \colon M_0 = M \to M_\alpha = \mathrm{Ult}_\alpha(M, E)$. In particular M_Ω is the result of iterating E out of the universe, so that $i_\Omega(\kappa) = \Omega$.

Let $\kappa = \mathrm{crit}(E)$. We write $\kappa_\nu = i_\nu(\kappa)$ and $I = \{\kappa_\nu \mid \nu \in \Omega\}$. We say that an ordinal β is a *generator belonging to* κ_ν if $\beta = i_\nu(\bar\beta)$ for some $\bar\beta \in [\kappa, \kappa^{+\omega_1})$

Note that the set of generators belonging to κ_ν is a subset of $\operatorname{supp}(i_\nu(E))$, that is, it is a set of generators for the extender $i_\nu(E)$ on κ_ν in M_ν. Every member of M_Ω is equal to $i_\Omega(f)(\vec{\beta})$ for some function $f \in M$ with domain $\kappa^{|\beta|}$ and some finite sequence $\vec{\beta}$ of generators for members of I. The following observation follows from this fact together with Proposition 3.1:

Proposition 3.3. *Suppose that $N \supseteq M_\Omega|\Omega$ is a model of set theory which contains all countable sets of generators. Then $\mathbb{C}^N = \mathbb{C}$.*

Proof. It is sufficient to show that N contains all countable sets of ordinals, but that is immediate since every countable set B of ordinals has the form $B = \{\, i_\Omega(f_n)(\vec{\beta}_n) \mid n \in \omega \,\}$, where each f_n is a function in M and each $\vec{\beta}_n$ is a finite sequence of generators. Since the sequence $\langle\, f_n \mid n \in \omega \,\rangle$ is in $M \subseteq N$ by Proposition 3.1, the sequence $\langle\, i_\Omega(f_n){\restriction}\lambda \mid n \in \omega \,\rangle \in M_\Omega|\Omega \subseteq N$ for $\lambda > \sup \bigcup_{n \in \omega} \vec{\beta}_n$, and the sequence $\langle\, \vec{\beta}_n \mid n \in \omega \,\rangle$ is in N by assumption. Thus $B \in N$. $\square$

Clearly the class I gives a sharp for the model $M_\Omega|\Omega$ in the sense of Definition 1.3 (with suitable sequences from I replaced by finite sequences), but it is not at all clear that I gives a sharp for $\mathbb{C}$ as well. We show starting in Section 3.3 that it does give a sharp when defined using the mouse specified there.

Conjecture 3.4. If M is the minimal mouse for which this procedure yields a sharp for $\mathbb{C}$, then the core model $K(\mathcal{R})^{\mathbb{C}}$ of the Chang model is given by an iteration k, without drops, of $M_\Omega|\Omega$.

This mouse M (which we will refer to as the "optimal" mouse) would then give "the" sharp for $\mathbb{C}$. A verification of this conjecture would presumably determine the correct large cardinal strength of the sharp, and remove some of the weaknesses which have been remarked on in our results.

3.1 Why is suitability required?

Two major weaknesses of the results of this paper were pointed out earlier: the need for restricted formulas and suitable sequences. We expressed the hope that the need for restricted formulas will be eliminated by strengthening these results to use the minimal mouse. In this subsection we make a brief digression to look at the question of suitability. Nothing in this subsection is required for the proof of Theorem 1.5(2) and nothing in this subsection will be referred to again except for the statement of Theorem 3.8.

Say that a mouse M is *correct for the Chang model* if there is an iteration $k\colon M_\Omega \to K(\mathcal{R})^{\mathbb{C}}$, without drops, such that $k[\kappa_\nu] \subset \kappa_\nu$ for all $\nu \in \Omega$ and $k(\kappa_\nu) > \kappa_\nu$ for all $\nu \in \Omega$ of uncountable cofinality.

Such a mouse must be the minimal mouse which is not a member of $\mathbb{C}$, since otherwise the minimal such mouse would be a member of M and the iteration k would either drop or go beyond Ω. The converse is not known, but it seems

probable that the minimal mouse is correct and that $i \restriction I = \{ (\kappa_\nu, k(\kappa_\nu) \mid \nu \in \Omega \}$ is a class of indiscernibles for $\mathbb{C}$.

Now suppose that M is correct for $\mathbb{C}$, and say that a sequence $\vec{\alpha}$ is Prikry for $\vec{\beta}$ if each is an increasing ω sequence and there is a sequence of measures $U_n \in M_\Omega$ on β_n such that $\vec{\alpha}$ satisfies the Mathias genericity condition: for all $x \subseteq \sup(\vec{\beta})$ in M_Ω, for all but finitely many $n \in \omega$, we have $\alpha_n \in x$ if and only if $x \cap \beta_n \in U_n$.

Note that we are not asserting here that $\vec{\alpha}$ is actually generic over M_Ω, as neither $\vec{\beta}$ nor the sequence of measures need be in M_Ω.

We write $\vec{\lambda} <^* \vec{\eta}$ if $\lambda_n < \eta_n$ for all but finitely many n.

Proposition 3.5. *Suppose $\vec{\nu}$ and $\vec{\mu}$ are increasing ω-sequences of ordinals with $\vec{\nu} <^* \vec{\mu}$ and $\sup(\vec{\nu}) = \sup(\vec{\mu})$. Then $\langle\, k(\kappa_{\nu_n}) \mid n \in \omega \,\rangle$ and $\langle\, \kappa_{\nu_n+1} \mid n \in \omega \,\rangle$ are each Prikry for $\langle\, k(\kappa_{\mu_n}) \mid n \in \omega \,\rangle$. Furthermore, no sequence $\vec{\alpha}$ in the interval $\langle\, k(\kappa_{\nu_n}) \mid n \in \omega \,\rangle <^* \vec{\alpha} <^* \langle\, \kappa_{\nu_n+1} \mid n \in \omega \,\rangle$ is Prikry for $\langle\, k(\kappa_{\mu_n}) \mid n \in \omega \,\rangle$.*

Proof. To see that $\langle\, \kappa_{\nu_n+1} \mid n \in \omega \,\rangle$ is Prikry for $\langle\, k(\kappa_{\mu_n}) \mid n \in \omega \,\rangle$, use $U_n = k \circ i_{\nu_n+1,\mu_n}(U_n')$ where $U_n' = \{\, x \subseteq \kappa_{\nu_n+1} \mid \kappa_{\xi_n+1} \in k(x) \,\}$. To see that $\langle\, k(\kappa_{\nu_n}) \mid n \in \omega \,\rangle$ is Prikry for $\langle\, k(\kappa_{\mu_n}) \mid n \in \omega \,\rangle$, use $U_n = k \circ i_{\mu_n}((E)_\kappa)$.

For the final sentence, observe that $\langle\, k \circ i_\Omega(f)(k(\kappa_\nu)) \mid f \in M \,\rangle$ is cofinal in $\kappa_{\nu+1}$ for all $\nu \in \Omega$. It follows that if $\langle\, k(\kappa_{\nu_n}) \mid n \in \omega \,\rangle <^* \vec{\alpha} <^* \langle\, \kappa_{\nu_n+1} \mid n \in \omega \,\rangle$ then there is a function $f \in M$ such that $k \circ i_\Omega(f)(k(\kappa_{\nu_n})) > \alpha_n$ for all $n \in \omega$ such that $\alpha_n < \kappa_{\nu_n+1}$, so $x = \{\, \nu \mid (\exists \nu' < \nu)\ k \circ i_\Omega(f)(\nu') > \nu \,\}$ witnesses that $\vec{\alpha}$ is not Prikry for $\langle\, \kappa_{\mu_n} \mid n \in \omega \,\rangle$. $\qquad\square$

Corollary 3.6. *Suppose that B and B' are two countable closed subsets of I such that for all formulas φ of set theory (with no extra terms) $\mathbb{C} \models \varphi(k \restriction B) \iff \varphi(k \restriction B')$.*

Then, writing $B = \langle\, \kappa_{\nu_\xi} \mid \xi < \alpha \,\rangle$ and $B' = \langle\, \kappa_{\nu_\xi'} \mid \xi < \alpha' \,\rangle$, we have $\alpha = \alpha'$, $(\forall \xi < \alpha)\ (\mathrm{cf}(\nu_\xi) = \omega \iff \mathrm{cf}(\nu_\xi') = \omega)$, and for all but finitely many $\xi < \alpha$

1. *$\nu_{\xi+1} = \nu_\xi + 1$ if and only if $\nu_{\xi+1}' = \nu_\xi' + 1$, and*

2. *ν_ξ is a limit ordinal if and only if ν_ξ' is a limit ordinal.*

Proof. Only the two numbered assertions are problematic. For the first assertion, suppose to the contrary that $\langle\, \xi_n \mid n \in \omega \,\rangle$ is an increasing subsequence of α such that $\nu_{\xi_n+1} = \nu_{\xi_n} + 1$ but $\nu_{\xi_n+1}' > \nu_{\xi_n}' + 1$. Let $\varphi(k \restriction B)$ be the formula asserting that there is no sequence $\vec{\alpha}$ which is Prikry for $\langle\, k(\kappa_{\nu_{\xi_n}+1}) \mid n \in \omega \,\rangle$ such that $\langle\, \kappa_{\nu_{\xi_n}} \mid n \in \omega \,\rangle <^* \vec{\alpha} <^* \langle\, \kappa_{\nu_{\xi_n}+1} \mid n \in \omega \,\rangle$ for each $n \in \omega$. Then φ is true of B but false of B'.

For the second assertion, observe that ν_{ξ_n} is a limit ordinal for all but finitely many $n \in \omega$ if and only if there are $<^*$-cofinally many sequences $\vec{\alpha} <^* \langle\, \kappa_{\nu_{\xi_n}} \mid n \in \omega \,\rangle$ which are Prikry for $\langle\, k(\kappa_{\nu_{\xi_n}}) \mid n \in \omega \,\rangle$. $\qquad\square$

On its face this Corollary is vacuous: it applies only to (and only conjecturally to) the optimal sharp for the Chang model, which itself only conjecturally exists. However it is an important motivation for the technique we use to prove

the Main Theorem and gives important information about the structure of the sharp of the Chang model. First, the *gaps* in a sequence B, that is, the maximal intervals of $I \backslash B$, are important. Second, (assuming as we do that no gaps have a least upper bound of cofinality ω) the only important characteristic of the gaps is whether their upper bound is a limit point or a successor point of I. Finally, individual gaps are not important—only infinite sets of gaps.

Indeed, in Subsection 4.8 we will outline a proof of Theorem 3.8 below, which strengthens Theorem 1.5(2) to show that the class I of indiscernibles of given by the proof of that theorem satisfies the converse of the conclusion of Corollary 3.6.

Definition 3.7. Call a sequence $B \subseteq I$ *weakly suitable* if B is a countable and closed, and $B \cap \lambda$ is unbounded in λ whenever $\lambda \in B$ and $\operatorname{cf}(\lambda) = \omega$.

Suppose that $B = \langle \lambda_\nu \mid \nu < \alpha \rangle$ and $B' = \langle \lambda'_\nu \mid \nu < \alpha' \rangle$, enumerated in increasing order, are weakly suitable. We say that B and B' are *equivalent* if $\alpha = \alpha'$, $(\forall \nu < \alpha)$ $(\operatorname{cf}(\lambda_\nu) = \omega \iff \operatorname{cf}(\lambda'_\nu) = \omega)$, and with at most finitely many exceptions the following hold for all $\nu < \alpha$: (i) $\lambda_{\nu+1} = \min(I \backslash \lambda_\nu + 1)$ if and only if $\lambda'_{\nu+1} = \min(I \backslash \lambda_\nu + 1)$, and (ii) λ_ν is a limit member of I if and only if λ'_ν is a limit member of I.

Theorem 3.8. *If B and B' are equivalent weakly suitable sequences then $\mathbb{C} \models \varphi(B) \iff \varphi(B')$ for any restricted formula φ in our language.*

3.2 Definition of the set T of terms.

The next definition gives the set of terms we will use to construct the sharp. This list should be regarded as preliminary, as a better understanding of the Chang model will undoubtedly suggest a more felicitous choice.

Definition 3.9. The members of the set T of terms of our language for the sharp of $\mathbb{C}$ are those obtained by compositions of the following set of basic terms:

1. For each function $f \colon {}^n\kappa \to \kappa$ in M for some $n \in \omega$, there is a term τ such that $\tau(z) = i_\Omega(f)(z)$ for all $z \in {}^n\Omega$.

2. For each $\bar{\beta}$ in the interval $\kappa \leqslant \bar{\beta} < (\kappa^{+\omega_1})^M$ there is a term τ such that $\tau(\kappa_\nu) = i_\nu(\bar{\beta})$ for all $\nu \in \Omega$.

3. Suppose $\langle \tau_n \mid n \in \omega \rangle$ is an ω-sequence of compositions of terms from the previous two cases, and $\operatorname{domain}(\tau_n) \subseteq {}^{k_n}\Omega_n$. Then there is a term τ such that $\tau(\vec{a}) = \langle \tau_n(\vec{a} \restriction k_n) \mid n \in \omega \rangle$ for all $\vec{a} \in {}^\omega\Omega$.

4. For each formula φ, there is a term τ such that if ι is an ordinal and y is a countable sequence of terms for members of $\mathbb{C}_\iota$ then

$$\tau(\iota, y) = \{\, x \in \mathbb{C}_\iota \mid \mathbb{C}_\iota \models \varphi(x, y) \,\}.$$

Proposition 3.10. *For each $z \in \mathbb{C}$ there is a term $\tau \in M$ and a suitable sequence B such that $\tau(B) = z$.*

Proof. First we observe that any ordinal ν can be written in the form $\nu = i_\Omega(f)(\vec{\beta})$ for some $f \in M$ and finite sequence $\vec{\beta}$ of generators. Each generator β belonging to some $\kappa_\xi \in I$ is equal to $i_\xi(\bar{\beta})$ for some $\bar{\beta} \in [\kappa, (\kappa^{+\omega_1})^M)$, and thus is denoted by a term $\tau(\kappa_\xi)$ built from clause 2. Thus any finite sequence of ordinals is denoted by an expression using terms of type 1 and 2. Since M is closed under countable sequences, adding terms of type 3 adds in all countable sequences of ordinals.

Finally, any set $x \in \mathbb{C}$ has the form $\{\, x \in \mathbb{C}_\iota \mid \mathbb{C}_\iota \models \varphi(x, y) \,\}$ for some ι, φ and y as in clause 4. Thus a simple recursion on ι shows that every member of $\mathbb{C}$ is denoted by a term from clause 4. $\qquad\square$

The terms of clause 2 force the limitation to restricted formulas in Theorem 1.5(2), since the domain of these terms is exactly the class I of indiscernibles. It is possible that a more natural set of terms would enable this restriction to be removed, but this would depend on a precise understanding of the iteration $k \colon M_\Omega \to K(\mathcal{R})^\mathbb{C}$ from Subsection 3.1.

Proposition 3.10 actually exposes a probable weakness in our current state of understanding of the Chang model. This proposition corresponds to the property of $0^\sharp$ that every ordinal α is definable is using as parameters members of the class I of indiscernibles. In the case of $0^\sharp$ this is only true if the parameters are allowed to include members of $I \setminus \alpha + 1$. In contrast, Proposition 3.10 says that α is alway denoted by a term $\tau(B)$ with $B \in [I \cap (\alpha + 1)]^\omega$. Possibly a more polished set of terms, obtained through a more careful analysis of the fine structure of the models and the iteration k, would yield definability properties more like those of $0^\sharp$.

3.3   Outline of the proof

Proposition 3.3 suggests a possible strategy for the proof of Theorem 1.5(2): find a generic extension of $M_\Omega | \Omega$ which contains all countable sequences of generators. There are good reasons why this is likely to be impossible, beginning with the problem of actually constructing a generic set for a class sized model. Beyond that, many of the known forcing constructions used to add countable sequences of ordinals require large cardinal strength far stronger than that assumed in the hypothesis of Theorem 1.5, and give models with properties which are known to imply the existence of submodels having strong large cardinal strength. However, two considerations suggest that this last problem may be less serious than it first appears. First, there can be much more large cardinal strength in the Chang model than is apparent from the actual extenders present in $K(\mathcal{R})^\mathbb{C}$, since much of the large cardinal strength in V is encoded in the set of reals. Second, many properties known to imply large cardinal strength are false in the Chang model not because of the lack of such strength, but because of the failure of the Axiom of Choice. Results involving the size of the power set of singular cardinals, for example, are irrelevant to the Chang model since the power set is not (typically) well ordered there.

We avoid the problem of constructing generic extensions for class sized model by working with submodels generated by countable subsets of I, and we find that in fact none of the large cardinal structure in V survives the passage to the Chang model beyond that given in the hypothesis to Theorem 1.5.

Definition 3.11. If $B \subseteq I$ and Gen_B is the set of generators belonging to members of B then we write

$$M_B = \{i_\Omega(f)(b) \mid f \in M \wedge b \in [\mathrm{Gen}_B]^{<\omega}\}.$$

If B is closed, and in particular if it is suitable, then we write $\mathbb{C}_B$ for the Chang model evaluated using the ordinals of $M_B|\Omega$ and all countable sequences of these ordinals.

Note that M_B is not transitive: it is a submodel of M_Ω, and $i_\Omega : M \to M_\Omega$ is the canonical embedding $M \to M_B$ for any $B \subseteq I$. It is not obvious even that the model $\mathbb{C}_B$ can be regarded as a subset of $\mathbb{C}$; the proof of this is a part of the proof of the main lemma. The definition of $\mathbb{C}_B$ does imply that if B and B' are closed subsets of I with the same order type then $\mathbb{C}_B \cong \mathbb{C}_{B'}$. In particular, if $\mathrm{otp}(B) = \alpha + 1$ then, setting $B(\alpha+1) = \{\kappa_\nu \mid \nu < \alpha + 1\}$, $\mathbb{C}_B \cong \mathbb{C}_{B(\alpha+1)}$, which in turn is equal to the $\kappa_{\alpha+1}$st stage $\mathbb{C}_{\kappa_{\alpha+1}}$ of the recursive definition of the Chang model as stated at the beginning of this paper.

The motivation for our work begins with the observation that $M_B|\Omega \prec M_{B'}|\Omega \prec M_\Omega|\Omega$ whenever $B \subseteq B' \subseteq I$. Corollary 3.6 refutes any suggestion that this necessarily extends to the models $\mathbb{C}_B$ and $\mathbb{C}_{B'}$, however it also motivates Definition 3.12 below.

Corollary 3.6 says that we must take account of the gaps in B. To be precise, we will say that a *gap* in B is a maximal nonempty interval in $I \backslash B$. For B either suitable or limit suitable, every gap in B is headed by a limit point λ of I which is a member of $B \cup \{\Omega\}$ and has uncountable cofinality.

Definition 3.12. A subset B of I is *limit suitable* if (i) its closure $\bar{B}$ is suitable, and every gap in B is an interval of the form $[\lambda, \delta)$ where (ii) δ is either Ω or a member of B which is a limit point of I of uncountable cofinality, (iii) if $\lambda \neq \varnothing$, then $\lambda = \sup(\{0\} \cup B \cap \delta)$, and (iv) $\lambda = \kappa_{\nu+\omega}$ for some $\nu \in \Omega$.

Two limit suitable sets B and B' are said to be *equivalent* if they have the same order type and they have gaps in the same locations. For a limit suitable set B, which is never closed (except for $B = \varnothing$), we write $\mathbb{C}_B = \bigcup\{\mathbb{C}_{B'} \mid B' \subset B \wedge B' \text{ is suitable}\}$. That is, for limit suitable sets B the model is constructed, like $\mathbb{C}_B$ for suitable B, by construction over the (nontransitive) set of ordinals of M_B, but using only those countable sets of ordinals which are in $\mathbb{C}_{B'}$ for some suitable $B' \subset B$.

The use of $\kappa_{\nu+\omega}$ in the final Clause (iv) is for convenience: our arguments would still be valid if it were only required that λ be a limit member of I of countable cofinality which is not a member of B.

Note that if B is a limit suitable sequence then $\mathbb{C}_B$ is not closed under countable sequences; in particular B is not a member of $\mathbb{C}_B$. Thus if δ is the head of a gap of B then $\mathbb{C}_B$ believes (correctly) that δ has uncountable cofinality.

Theorem 1.5(2) will follow from the following lemma:

Lemma 3.13 (Main Lemma). *Suppose $B \subset I$ is limit suitable. Then $\mathbb{C}_B$ is isomorphic to an elementary substructure of $\mathbb{C}$ via the map defined by $\tau^{\mathbb{C}_B}(\vec{\beta}) \mapsto \tau^{\mathbb{C}}(\vec{\beta})$ for any term $\tau \in T$ and any $\vec{\beta}$ which is a countable sequence of generators for members of some suitable $B' \subset B$.*

The elementarity holds for all restricted formulas. The proof will be by an induction over pairs (ι, φ), where $\iota \in M_B \cap \Omega + 1$, and φ is a formula of set theory; and the induction hypothesis implies that the map

$$\{\, z \in \mathbb{C}_\iota^{\mathbb{C}_B} \mid \mathbb{C}_\iota^{\mathbb{C}_B} \models \varphi(z, \vec{\beta}) \,\} \mapsto \{\, z \in \mathbb{C}_\iota \mid \mathbb{C}_\iota \models \varphi(z, \vec{\beta}) \,\}$$

is well defined. To see that Lemma 3.13 suffices to prove Theorem 1.5(2), observe that any suitable set B can be extended to a limit suitable set defined by the equation

$$B' = B \cup \{\, \kappa_{\nu+n} \mid \kappa_\nu \in B \wedge n \in \omega \,\},$$

that is, by by adding the next ω-sequence from I at the foot of each gap of B and to the top of B. Now let B_0 and B_1 be two equivalent suitable sets. Then their limit suitable extensions B_0' and B_1' are also equivalent, having the same order type and having gaps in the corresponding places, so $\mathbb{C}_{B_0'} \cong \mathbb{C}_{B_1'}$. Then for any restricted formula φ we have

$$\mathbb{C} \models \varphi(B_0) \iff \mathbb{C}_{B_0'} \models \varphi(B_0)$$
$$\iff \mathbb{C}_{B_1'} \models \varphi(B_1) \iff \mathbb{C} \models \varphi(B_1).$$

4 The Proof of the Main Lemma

At this point we fix a mouse M to be used for the proof of the Main Lemma 3.13. Some basic properties of M have already been sketched at the start of Section 3, and Definition 4.1 below gives more specific requirements.

For this section, $B \subseteq I$ is a limit suitable sequence and $\zeta = \mathrm{otp}(B)$. The main tool used for the proof is the forcing $P(\vec{E}\restriction\zeta)/\leftrightarrow$, to be defined inside M, and a M_B-generic set $G \subseteq i_\Omega(P(\vec{E}\restriction\zeta)/\leftrightarrow)$ to be constructed inside $V[h]$ for a generic Levy collapse map $h\colon \omega_1 \cong \mathcal{R}$. The model $M_B[G]$ will include all its countable subsets, and $\mathbb{C}_B$ will be definable as a submodel of $M_B[G]$.

The forcing is essentially due to Gitik (see, for example, [2]) and the technique for constructing the M_B-generic set G is from Carmi Merimovich [9]. Gitik's forcing was designed to make the Singular Cardinal Hypothesis fail at a cardinal of cofinality ω by adding many Prikry sequences, each of which is (in our context) a sequence of generators for cardinals in B. Thus it would do what we need for the case when $\mathrm{otp}(B) = \omega$, but needs to be adapted to work for sequences B of arbitrary countable length. To this end we modify Gitik's forcing by using ideas introduced by Magidor in [8] to adapt Prikry forcing in order to to add sequences of indiscernibles of length longer than ω. This adds some

complications to Gitik's forcing, but on the other hand much of the complication of Gitik's work is avoided since we do not need to know whether cardinals in the interval $(\kappa^+, \kappa^{+\omega_1})$ are collapsed, and hence we can omit his preliminary forcing.

Our forcing is based on a sequence $\vec{E}$ of extenders, derived from the last extender E of M. We begin by defining this sequence, and at the same time specify what properties we require of the chosen mouse M.

Definition 4.1. We define an increasing sequence, $\langle\, N_\nu \mid \nu < \omega_1 \,\rangle$ of submodels of M. We write E_ν for $E \cap N_\nu$, the restriction of E to the ordinals in N_ν, we write $\pi_\nu \colon \bar{N}_\nu \to N_\nu$ for the Mostowski collapse of N_ν, and we write $\bar{E}_\nu$ for $\pi_\nu^{-1}[E_\nu] = \pi_\nu^{-1}(E) \cap \bar{N}_\nu$.

We require that the $\mathcal{R}$-mouse M and the sequence $\langle\, N_\nu \mid \nu < \omega_1 \,\rangle$ satisfy the following conditions:

1. M is a model of Zermelo set theory such that $\mathcal{R} \subset M$, $|M| = |\mathcal{R}|$, and $\mathrm{cf}(\Omega \cap M) = \omega_1$.
2. $\mathrm{length}(E) = (\kappa^{+\omega_1})^M$.
3. If $\nu' < \nu < \omega_1$ then $(N_{\nu'}, E_{\nu'}) \prec (N_\nu, E_\nu) \prec (M, E)$.
4. $^{\kappa}N_\nu \cap M \subseteq N_\nu$.
5. $|\bar{N}_\nu|^M \subset N_\nu$.
6. $|\bar{N}_0|^M = (\kappa^{++})^M$, and if $\nu > 0$ then $|\bar{N}_\nu|^M = \sup_{\nu' < \nu}(|\bar{N}_{\nu'}|^{++})^M$.
7. $M = \bigcup_{\nu < \omega_1} N_\nu$.

Clauses 5 and 6 are needed for the proof of Proposition 4.40.

We will work primarily with the extenders E_ν rather than with their collapses $\bar{E}_\nu$, because this makes it easier to keep track of the generators. However it should be noted that E_ν may not be a member of $\mathrm{Ult}(M, E)$, so further justification is needed for many of the claims we wish to make about being able to carry out constructions inside M. Since we never actually use more than countably many of the extenders E_ν at any one time, the following observation will provide such justification:

Proposition 4.2. *The following are all members of* $\mathrm{Ult}(M, E_\nu)$, *for any* $\nu < \omega_1$:

- $\mathcal{P}(\bigcup_{\nu' < \nu} \bar{N}_{\nu'})$

- *the extender* $\bar{E}_{\nu'}$, *and the map* $\pi_{\nu''}^{-1} \circ \pi_{\nu'} \colon \mathrm{supp}(\bar{E}_{\nu'}) \to \mathrm{supp}(\bar{E}_{\nu''})$, *for each* $\nu' < \nu'' < \nu$

- *the direct limit of the set* $\{\, \mathrm{supp}(\bar{E}_{\nu'}) \mid \nu' < \nu'' < \nu \,\}$ *along the maps* $\pi_{\nu''}^{-1} \circ \pi_{\nu'}$, *as well as with the injection maps from* $\mathrm{supp}(E_{\nu'})$ *into this direct limit*

$\square$

Since $\mathrm{Ult}(M, E_\nu) = \mathrm{Ult}(M, \bar{E}_\nu)$, this proposition allows us to regard the direct limit as a code inside M for the extender E_ν together with its system of subextenders $E_{\nu'}$ for $\nu' < \nu$.

The hypothesis of Theorem 1.5 is more than sufficient to find a mouse M and sequence $\vec{N}$ of submodels satisfying Definition 4.1: this can be done by first defining models M' and $\langle\, N'_\nu \mid \nu < \omega_1 \,\rangle$ satisfying all of the conditions except Clause 7, and then taking M to be the transitive collapse of $\bigcup_{\nu < \omega_1} N'_\nu$. The conditions on M are, in turn, much stronger than is needed to carry out this construction. In view of the fact that there is no clear reason to believe that the actual strength needed is greater that $o(E) = \kappa^{+(\omega+1)}$, it does not seem useful to complicate the argument in order to determine the minimal mouse for which the present argument works.

We are now ready to begin the proof of Lemma 3.13. Following Gitik we define, in subsections 4.1 and 4.2, a Prikry type forcing $P(\vec{F})$ depending on a sequence $\vec{F}$ of extenders. Subsections 4.3 and 4.4 develop the properties of this forcing, and Subsection 4.5 describes an equivalence relation $\leftrightarrow$ on its set of conditions. Subsection 4.6 constructs an M_B-generic subset of $i_\Omega(P(\vec{E}\restriction\zeta)/\leftrightarrow)$, and subsection 4.7 uses this construction to prove Lemma 3.13 under the additional assumption that $\kappa = \kappa_0 \in B$. Finally, Subsection 4.8 deals with the special case $\kappa \notin B$ and indicates how the same technique can be used to prove Theorem 3.8.

4.1 The forcing $P(\vec{F})$

Throughout the definition of the forcing, from Subsections 4.1 through 4.5, we work entirely inside the mouse M; in particular all cardinal calculations are carried out inside M. We are interested in defining $P(\vec{E}\restriction\zeta)$, but for the purposes of the recursion used in the definition we allow $\vec{F}$ to be any suitable sequence of extenders. We will not give a definition of the notion of a *suitable sequence* of extenders. All the sequences used in this section are suitable: specifically, all of the sequences $\vec{E}\restriction\xi$ for $\xi < \omega_1$ are suitable, all of the ultrafilters $(E)_{\vec{E}\restriction\xi} = \{\, X \subseteq H_\kappa^M \mid \vec{E}\restriction\xi \in i^E(X) \,\}$ concentrate on suitable sequences, and furthermore, if $\vec{F}$ is suitable then so is $\vec{F}\restriction[\gamma_0, \tau)$ for any $0 \leqslant \gamma_0 \leqslant \tau \leqslant \mathrm{length}(\vec{F})$.

Before starting the definition of the forcing, we give a brief discussion of its design, techniques and origin.

The constructed generic extension of M_B will have the form $M[G] = M[\vec{\kappa}, \vec{h}]$, where $\vec{\kappa} = \langle\, \bar{\kappa}_\gamma \mid \gamma \leqslant \zeta \,\rangle$ enumerates $B \cup \{\Omega\}$ and $\vec{h} = \langle\, h_{\nu,\nu'} \mid \zeta \geqslant \nu > \nu' \,\rangle$ is a sequence of functions $h_{\nu,\nu'} \colon [\bar{\kappa}_\nu, \bar{\kappa}_\nu^+) \to \bar{\kappa}_\nu$. Each of the functions $h_{\nu,\nu'}$ is, individually, Cohen generic over M.

The purpose of this forcing is to provide what we will call "standard forcing names" for the generators belonging to members of B. Specifically, consider $\Omega = \kappa_\Omega \in M_B$ and suppose $\beta = i_{\bar\nu}(\bar\beta)$ is a generator belonging to $\kappa_{\bar\nu} = \bar\kappa_\nu \in B$. The construction of the M_B-generic set G will determine an ordinal $\bar\xi \in [\kappa, \kappa^+)$ such that $\beta = h_{\zeta,\nu}(i_\Omega(\bar\xi))$, and this will be used as a name in M, with parameters ν and $\bar\xi$, for the generator β in M_B. Since M is closed under countable sequences, this will give a name for any countable sequence of generators, and this in turn will give, via clause 4 of Definition 3.9, a name for any member of $\mathbb{C}_B$.

The problem comes from the fact that the forcing $P(\vec{E}\restriction\zeta)$ only uses the extenders E_ν for $\nu < \zeta$. The raw use of the iteration $\langle\, i_\xi \mid \xi \in \Omega \,\rangle$ would specify

that $i_\Omega(\bar\beta)$, for $\bar\beta \in [\kappa, \kappa^+)$, should be assigned the indiscernibles $\{\, i_{\bar\nu}(\bar\beta) \mid \kappa_{\bar\nu} = \bar\kappa_\nu \in B \,\}$; however this would establish names only for the generators $i_{\bar\nu}(\bar\beta)$ such that $\bar\beta \in \bigcup_{\nu < \zeta} \operatorname{supp}(E_\nu)$. To get around this problem we need to have a way to slip any ordinal $i_{\bar\nu}(\bar\beta)$, for $\kappa_{\bar\nu} = \bar\kappa_\nu \in B$ and $\bar\beta \in [\kappa, \kappa^{+\omega_1})$, into the generic set as a substitute for some $i_{\bar\nu}(\bar\beta')$ with $\bar\beta' \in \bigcup_{\nu < \zeta} \operatorname{supp}(E_\nu)$.

The trick is to design the forcing to disassociate the indiscernibles added by the Prikry component of the forcing from any particular ordinal for which it is an indiscernible. We follow Gitik [2, 3, 4, 5] in using three successive stages to do so.

The first stage involves mixing Cohen forcing in with the Prikry forcing. For any apparent indiscernible $h_{\gamma, \gamma'}(\xi) = \xi'$ determined by the generic set G, there are conditions in G which assign the value via a Cohen condition as well as conditions which assign it via a Prikry condition. In particular, there is no function in $M_B[G]$ which assigns uniform indiscernibles to any subset of $[\kappa_\Omega, \kappa_\Omega^{+\omega_1})$ of size greater than $\Omega = \kappa_\Omega$.

The second stage involves the use of $[\kappa_\Omega, \kappa_\Omega^+)$ as the domain of $h_{\zeta, \nu}$, rather than $\bigcup_{\nu < \zeta} \operatorname{supp}(i_\Omega(E_\nu))$. This is accomplished by using, in the Prikry component of the forcing, functions $a = a_{\zeta, \nu}^{s, \zeta}$ which map a subset of $[\kappa_\Omega, \kappa_\Omega^+)$ of size Ω into $\operatorname{supp}(i_\Omega(E_\nu))$. The atomic non-direct extension will use a function a', taken from a member of the ultrafilter $(i_\Omega(E_\nu))_a$. The function a' could be regarded as a Prikry indiscernible for a; however it will be recorded in the extension only via a Cohen condition $f_{a, a'}$ defined by $f(\xi) = a'(\xi')$, where $\xi' \in \operatorname{domain}(a')$ corresponds to $\xi \in \operatorname{domain}(a)$.

The effect of this is that if $\alpha \in i_\Omega(\operatorname{supp}(E_0))$ and s is a condition including $a_{\zeta, \nu}^{s, \zeta}(\xi) = \alpha$ for each $\nu < \zeta$, then the sequence $\vec\beta = \langle\, h_{\zeta, \nu}(\xi) \mid \nu < \zeta \,\rangle$ in $M_B[G]$ will be a Prikry sequence for the ultrafilter $(i_\Omega(E_0))_\alpha$; however there will be no association, or at least no explicit association, with the ordinal β as distinguished from any other member of $\{\, \beta' \in [\kappa_\Omega, \kappa_\Omega^{+\omega_1}) \mid (i_\Omega(E_0))_{\beta'} = (i_\Omega(E_0))_\beta \,\}$, which will for typical β be unbounded in $\operatorname{supp}(i_\Omega(E_\nu))$ for each $\nu \leqslant \zeta$.

The ambiguity introduced by the second stage allows the third, and final, stage in the disassociation of the Prikry conditions, via the equivalence relation $\leftrightarrow$ introduced in Subsection 4.5. Gitik uses this equivalence relation to ensure that the final forcing has the κ^{++}-chain condition and hence does not collapse κ^{++}. We do not care whether the cardinals $\bar\kappa_\nu^{++}$ are collapsed in $M_B[G]$, but we need to use the equivalence relation in order to construct a generic set G which gives standard forcing names to all generators $i_{\bar\nu}(\bar\beta)$ belonging to $\bar\kappa_\nu = \kappa_{\bar\nu} \in B$. This may be regarded as a way of making the notions of "no association" versus "no explicit association" in the last paragraph more precise. As an example of a non-explicit association, suppose that $(E)_{\beta'} \neq (E)_\beta$ for all $\beta' < \beta$. Then E_β is necessarily associated with the least of the Prikry sequences for the ultrafilter $(E)_\beta$. Thus, in this case, the association, though not explicit, is unavoidable. The equivalence relation $\leftrightarrow$ will allow us to determine, for any ordinal $\bar\beta \in [\kappa, \kappa^{+\omega_1})$, sequences $\langle \bar\beta_\nu \mid \nu < \zeta \rangle$ with $\bar\beta_\nu \in \operatorname{supp}(E_\nu)$ such that the Prikry sequence $\langle\, i_\nu(\bar\beta) \mid \kappa_\nu \in B \,\rangle$ induced by the iteration i can be substituted in the constructed generic set for the sequence $\langle\, i_\nu(\bar\beta_\nu) \mid \kappa_\nu \in B \,\rangle$ which would be

assigned by the iteration i_Ω as the indiscernibles associated with $\langle\, i_\Omega(\bar\beta_\nu) \mid \nu < \zeta\,\rangle$.

4.1.1 Definition of the forcing: Overview

Definition 4.3. The conditions of $P(\vec F)$ are functions s satisfying the following conditions:

1. The domain of s is a finite subset of $\zeta + 1$ with $\zeta \in \operatorname{domain}(s)$.

2. Each value $s(\tau)$ of s is a member of the set P_τ^* of quadruples

$$s(\tau) = (\bar\kappa^{s,\tau}, \vec F^{s,\tau}, z^{s,\tau}, \vec A^{s,\tau}).$$

   satisfying the following conditions:

   (a) $\vec F^{s,\tau}$ is a suitable sequence $\vec F^{s,\tau} = \langle\, F_\nu^{s,\tau} \mid \gamma_0 \leqslant \nu < \tau\,\rangle$ of extenders, where $\gamma_0 = \max(\operatorname{domain}(s)\cap\tau)+1$, or $\gamma_0 = 0$ if $\tau = \min(\operatorname{domain}(s))$.

   (b) $\bar\kappa^{s,\tau}$ is the critical point of the extenders in $\vec F^{s,\tau}$.

   (c) $z^{s,\tau}$ is a tableau of functions giving information about the functions $h_{\nu,\nu'}$. This tableau will be fully specified in Definition 4.4.

   (d) $\vec A^{s,\tau}$ is a sequence of sets $A_\nu^{s,\tau} \in U_\nu^{s,\tau}$, for $\gamma_0 \leqslant \nu < \tau$. The definition of the ultrafilter $U_\nu^{s,\tau}$ will be given in Definition 4.5.

The two partial orders on $P(\vec F)$, a direct extension order $\leqslant^*$ and a forcing order $\leqslant$, will be defined in Subsection 4.2.

4.1.2 Definition of the forcing: the tableau $z = z^{s,\tau}$

The third component $z^{s,\tau}$ of $s(\tau)$ is a tableau which is represented in Figure 1.
   The following definition specifies the members of this tableau:

Definition 4.4. Suppose that $\tau \in \operatorname{domain}(s)$, and set $\gamma_0 = \sup(\operatorname{domain}(s) \cap \tau) + 1$, or $\gamma_0 = 0$ if $\tau = \min(\operatorname{domain}(s))$. The tableau $z = z^{s,\tau}$ includes

1. for each pair (γ, ν) of ordinals with $\tau \geqslant \gamma \geqslant \gamma_0 > \nu \geqslant 0$, a function $f_{\gamma,\nu}^z$ and

2. for each pair (γ, ν) with $\tau \geqslant \gamma > \nu \geqslant \gamma_0$, a pair of functions $(a_{\gamma,\nu}^z, f_{\gamma,\nu}^z)$.

For each pair γ, ν the function $f_{\gamma,\nu}^z = f_{\gamma,\nu}^{s,\tau}$ is a slightly modified Cohen function:

1. $\operatorname{domain}(f_{\gamma,\nu}^z) \subseteq [\bar\kappa^z, (\bar\kappa^z)^+)$ and $|\operatorname{domain}(f_{\gamma,\nu}^z)| \leqslant \bar\kappa^z$.

2. Each of the values $f_{\gamma,\nu}^z(\xi)$ of $f_{\gamma,\nu}^z$ has one of the two following forms:

   (a) $f_{\gamma,\nu}^z(\xi) = \xi' \in \bar\kappa_\tau^z$, or

   (b) $f_{\gamma,\nu}^z(\xi) = h_{\gamma',\nu}(\xi')$ for some γ' in the interval $\gamma > \gamma' > \nu$ and some $\xi' \in \bar\kappa_\tau^z$.

	0	$\cdots$	$\gamma_0 - 1$	γ_0	$\cdots$	γ	$\cdots$
τ	$f^z_{\tau,0}$	$\cdots$	f^z_{τ,γ_0-1}	$(a^z_{\tau,\gamma_0}, f^z_{\tau,\gamma_0})$	$\cdots$	$(a^z_{\tau,\gamma}, f^z_{\tau,\gamma})$	$\cdots$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	
γ	$f^z_{\gamma,0}$	$\cdots$	f^z_{γ,γ_0-1}	$(a^z_{\tau,\gamma_0}, f^z_{\gamma,\gamma_0})$	$\cdots$		
$\vdots$	$\vdots$		$\vdots$	$\vdots$			
$\gamma_0 + 1$	$f^z_{\gamma_0+1,0}$	$\cdots$	$f^z_{\gamma_0+1,\gamma_0-1}$	$(a^z_{\gamma_0+1,\gamma_0}, f^z_{\gamma_0+1,\gamma_0})$			
γ_0	$f^z_{\gamma_0,0}$	$\cdots$	$f^z_{\gamma_0,\gamma_0-1}$				

Figure 1: The third component $z^{s,\tau}$ of $s(\tau)$. The element at row α and column β is used to determine $h_{\alpha,\beta}$. In the case of the top row, this determination is direct; for the other rows this is indirect, via their use in defining the ultrafilters $U^{s,\tau}_\gamma$ from which the sets $A^{s,\tau}_\alpha$ are taken.

The functions $a^z_{\gamma,\nu} = a^{s,\tau}_{\gamma,\nu}$ satisfy the following conditions:

1. $\operatorname{domain}(a^z_{\gamma,\nu}) \subseteq [\bar\kappa^z, (\bar\kappa^z)^+)$ and $|\operatorname{domain}(a^z_{\gamma,\nu})| \leqslant \bar\kappa^z$.

2. $\operatorname{range}(a^z_{\gamma,\nu}) \subseteq \operatorname{supp}(F^{s,\tau}_\nu)$.

3. $\operatorname{domain}(a^z_{\gamma,\nu}) \cap \operatorname{domain}(f^z_{\gamma,\nu}) = \varnothing$,

4. If $\tau \geqslant \gamma > \gamma' > \nu$ then $a^z_{\gamma,\nu} \subseteq a^z_{\gamma',\nu}$.

The (γ, ν) entry in the tableau, whether a function $f^z_{\gamma,\nu}$ or a pair of functions $(a^z_{\gamma,\nu}, f^z_{\gamma,\nu})$, will ultimately be used to determine the values of the Cohen function $h_{\gamma,\nu}$.

The functions $f^{s,\tau}_{\tau,\nu}$ in the first row of z directly determine $h_{\tau,\nu}$. The functions $f^{s,\tau}_{\gamma,\nu}$ in the remaining rows, with $\gamma < \tau$, indirectly help to determine $h_{\gamma,\nu}$ via the Prikry style forcing: they restrict the possible values of $s'(\gamma)$ in conditions $s' \leqslant s$.

The first form for the function $f_{\gamma,\nu}$ is the usual form for a Cohen condition and asserts that $h_{\gamma,\nu}(\xi) = \xi'$; or, more specifically, if s is a condition with $f^{s,\tau}_{\tau,\gamma}(\xi) = \xi'$, then $s \Vdash \dot h_{\tau,\gamma}(\xi) = \xi'$. The second form, the value $f^{s,\tau}_{\tau,\gamma}(\xi) = h_{\gamma',\nu}(\xi')$, of $f(\xi)$ may be taken as a formal expression: it specifies that the value of the name $h_{\tau,\nu}(\xi)$ is given by

$$\text{if } s \Vdash \dot h_{\gamma',\nu}(\xi') = \xi'' \text{ then} \qquad s \Vdash \dot h_{\tau,\nu}(\xi) = \xi'', \tag{1}$$
$$\text{if } s \Vdash \xi' \notin \operatorname{domain}(\dot h_{\gamma',\nu}) \text{ then} \qquad s \Vdash \dot h_{\tau,\nu}(\xi) = 0, \text{ and}$$
$$\text{otherwise} \qquad s \nVdash \dot h_{\tau,\nu}(\xi).$$

This definition requires recursion on τ, using the fact that "$s \Vdash \dot h_{\gamma',\nu}(\xi') = \xi''$" depends only on $s{\restriction}\gamma' + 1$. In the first of these three cases, $s \Vdash \dot h_{\gamma',\nu}(\xi') = \xi''$, we will regard the forms $f^z_{\tau,\nu}(\xi) = \xi''$ and $f^z_{\tau,\nu}(\xi) = h_{\gamma',\nu}(\xi')$ as being identical.

The functions $a_{\gamma,\nu}^z$ are included in order to generate the Prikry indiscernibles. If $a_{\tau,\nu}^{s,\tau}(\xi) = \alpha$, then $h_{\tau,\nu}(\xi)$ in the generic extension will be a Prikry indiscernible for the ultrafilter $(F_\nu^{s,\tau})_\alpha = \{\, x \in \mathcal{P}(\bar\kappa) \mid \alpha \in i_\nu^{F^{s,\tau}}(x) \,\}$.

This completes the definition of the tableau $z^{s,\tau}$.

4.1.3 The forcing: the ultrafilters $U_\gamma^{s,\tau}$ and sets $A_\gamma^{s,\tau}$.

We continue the definition of $P(\vec{F})$ by specifying the requirements for the final coordinate $A^{s,\zeta}$ for a quadruple $w = s(\zeta) \in P_\zeta^*$. Definition 4.5 uses recursion on ζ to define the following for each for $\gamma < \zeta$:

1. a set $P_{\zeta,\gamma}^*$, of which A_γ^w is a subset,

2. a restriction operation $w{\uparrow}\gamma$, which maps $w \in P_\zeta^*$ to a quadruple $w{\uparrow}\gamma \in P_{\zeta,\gamma}^*$, and

3. an ultrafilter $U_\gamma^w \subset \mathcal{P}(P_{\zeta,\gamma}^*)$.

These will complete the definition of the set $P_\gamma^* = P_{\gamma,\gamma}^*$, and hence of the set of conditions of the forcing $P(\vec{F})$.

In addition to $w{\uparrow}\gamma$ we use a second restriction operator $z{\restriction}[\gamma_0, \gamma]$, which may be applied to a tableau z of the form of either Figure 1 or 2. This operator retains the rows of z with indices in the interval $[\gamma_0, \gamma]$ and discards the rows above these; thus if $w = (\bar\kappa^w, \vec{F}^w, z^w, \vec{A}^w) \in P_\zeta^*$, then $(\bar\kappa^w, \vec{F}^w{\restriction}\gamma, z^w{\restriction}[\gamma_0, \gamma], \vec{A}^w{\restriction}\gamma) \in P_\gamma^*$.

Definition 4.5. We assume as a recursion hypothesis that P_τ^* and $P_{\tau,\gamma}^*$ have been defined for all $\gamma \leqslant \tau < \zeta$. If $\zeta \geqslant \gamma$ then the members of $P_{\zeta,\gamma}^*$ are quadruples

$$w = (\bar\kappa^w, \vec{F}^w, z^w, \vec{A}^w) \tag{2}$$

satisfying the following conditions:

1. The tableau z^w has the form of Figure 2.

2. $w{\restriction}[\gamma_0, \gamma] = (\bar\kappa^w, \vec{F}^w, z^w{\restriction}[\gamma_0, \gamma], \vec{A}^w) \in P_\gamma^*$.

3. The functions $a_{\nu,\nu'}^z$ for $\tau \geqslant \nu > \gamma \geqslant \nu'$ satisfy the conditions in Definition 4.4, except that $a_{\tau,\nu'}^z$ has range contained in $[\bar\kappa_\tau, (\bar\kappa_\tau)^{+\omega_1})$.

Note that $P_{\tau,\tau}^* = P_\tau^*$.

Suppose that $\tau \leqslant \zeta$, $w \in P_{\tau,\gamma}^*$ and $\gamma' < \gamma$. Then $w{\uparrow}\gamma'$ is the quadruple

$$w{\uparrow}\gamma' = (\bar\kappa^w, \vec{F}^w{\restriction}\gamma', z^w{\uparrow}\gamma', \vec{A}^w{\uparrow}\gamma') \in P_{\tau,\gamma'}^*$$

defined by recursion on γ as follows:

1. $z^w{\uparrow}\gamma'$ is equal to the tableau obtained by deleting from z^w all columns with index greater than γ' and deleting the functions $f_{\nu,\nu'}^z$ from all rows with index greater than γ'. Thus $(z^w{\uparrow}\gamma'){\restriction}[\gamma_0, \gamma'] = z{\restriction}[\gamma_0, \gamma']$ but the rows with index $\nu > \gamma'$ retain only the functions $a_{\nu,\nu'}^w$ for $\gamma_0 \leqslant \nu' < \nu \leqslant \gamma$.

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	0	$\cdots$	γ_0-1	γ_0	$\cdots$	γ
τ				a^z_{τ,γ_0}	$\cdots$	$a^z_{\tau,\gamma}$
$\vdots$				$\vdots$		$\vdots$
$\gamma+1$				$a^z_{\gamma+1,\gamma_0}$	$\cdots$	$a^z_{\gamma+1,\gamma}$
γ	$f^z_{\gamma,0}$	$\cdots$	f^z_{γ,γ_0-1}	$(a^z_{\gamma,\gamma_0}, f^z_{\gamma,\gamma_0})$	$\cdots$	
$\vdots$	$\vdots$		$\vdots$	$\vdots$		
γ_0+1	$f^z_{\gamma_0+1,0}$	$\cdots$	$f^z_{\gamma_0+1,\gamma_0-1}$	$(a^z_{\gamma_0+1,\gamma_0}, f^z_{1,\gamma_0})$		
γ_0	$f^z_{\gamma_0,0}$	$\cdots$	$f^z_{\gamma_0,\gamma_0-1}$			

Figure 2: The tableau z^w of a member of $A^{s,\tau}_\gamma \subseteq P^*_{\tau,\gamma}$. The entry in row α and column β is used in the determination of $h_{\alpha,\beta}$.

2. $\vec{A}^w{\uparrow}\gamma' = \langle\, A^w_{\gamma''}{\uparrow}\gamma' \mid \gamma_0 \leqslant \gamma'' \leqslant \gamma' \,\rangle$ where $A^w_{\gamma''}{\uparrow}\gamma' = \{\, w'{\uparrow}\gamma' \mid w' \in A^w_{\gamma''} \,\}$. Note that this definition also applies for $w \in P^*_\tau$, since $P^*_\tau = P^*_{\tau,\tau}$. Finally, the ultrafilter $U^{s,\tau}_\gamma$ is defined as

$$U^{s,\tau}_\gamma = (F^{s,t}_\gamma)_{s(\tau){\uparrow}\gamma} = \{\, X \subseteq P^*_{\tau,\gamma} \mid s(\tau){\uparrow}\gamma \in i^{F^{s,\tau}_\gamma}(X) \,\}. \tag{3}$$

This completes the definition of the set of conditions for the forcing $P(\vec{F})$.

4.2 The partial orderings of $P(\vec{F})$.

Since $P(\vec{F})$ is a Prikry type forcing notion, we need to define both a direct extension order $\leqslant^*$ and a forcing order $\leqslant$. We will begin by defining the one-step extension, $\mathrm{add}(s,w) \leqslant s$, which is the atomic extension adding a new ordinal to the domain of s. We will then define the direct extension order $\leqslant^*$, which will be the restriction of $\leqslant$ to conditions $s' \leqslant s$ with $\mathrm{domain}(s') = \mathrm{domain}(s)$. The forcing extension $\leqslant$ is then the smallest transitive relation extending $\leqslant^*$ such that $\mathrm{add}(s,w) \leqslant s$ for all $w \in \bigcup_{\tau\in\mathrm{domain}(s)} \bigcup_\gamma A^{s,\tau}_\gamma$.

4.2.1 The one-step extension

The one-step extension $s' = \mathrm{add}(s,w)$ in $P(\vec{F})$ is the atomic non-direct extension, corresponding to the extension in Prikry forcing which simply adds one new ordinal to the finite sequence. In $P(\vec{F})$ it acts by merging Prikry components $a^{s,\tau}_{\nu,\nu'}$ of $s(\tau)$ into the corresponding Cohen components of $s'(\tau)$. The following preliminary definition specifies the conversion of $a^{s,\tau}_{\nu,\nu'}$ to a Cohen condition.

Definition 4.6. Suppose $w \in A^{s,\tau}_\gamma$ and $\tau \geqslant \nu > \gamma \geqslant \nu' \geqslant \gamma_0$, and let $a = a^{s,\tau}_{\nu,\nu'}$ and $a' = a^w_{\nu,\nu'}$. The Cohen condition $f_{a,a'}$ is defined as follows:

First, we define, for any function a with domain a set of ordinals, a map $\sigma_{a,r} \colon |\operatorname{domain}(a)| \cong \operatorname{domain}(a)$.[2] Write φ_a for the least Σ_0 formula, with ordinal parameters, such that for some $r \in \mathcal{R}$ the equation

$$\sigma_{a,r}(\nu) = \xi \iff \varphi_a(r, \nu, \xi) \tag{4}$$

defines an enumeration $\sigma_{x,r} \colon |\operatorname{domain}(a)| \cong \operatorname{domain}(a)$, and write R_a for the set of $r \in \mathcal{R}$ such that this holds.

If $r \in R_a \cap R_{a'}$ then $f_{a,a',r}$ is the Cohen condition defined by

$$f_{a,a',r}(\xi) = \begin{cases} a'(\sigma_{a',r} \circ \sigma_{a,r}^{-1}(\xi)) & \text{if } \sigma_{a,r}^{-1}(\xi) < \bar{\kappa}^w \text{ and } \nu' = \gamma, \\ h_{\gamma,\nu'}(\sigma_{a',r} \circ \sigma_{a,r}^{-1}(\xi')) & \text{if } \sigma_{a,r}^{-1}(\xi) < \bar{\kappa}^w \text{ and } \nu' < \gamma, \\ 0 & \text{if } \sigma_{a,r}^{-1}(\xi) \geqslant \bar{\kappa}^w, \end{cases} \tag{5}$$

using in the second case the second form (2b) of the Cohen condition from Definition 4.3. Then $f_{a,a'}$ is defined if and only if $R_a = R_{a'}$ and $(\forall r, r' \in R_a)\ f_{a,a',r} = f_{a,a',r'}$, in which case $f_{a,a'}$ is this common value of $f_{a,a',r}$.

Proposition 4.7. *Suppose that F is an extender with critical point λ.*

1. *If $|\operatorname{domain}(a)| = \lambda$ then $\{\, a' \mid f_{a,a'} \text{ exists} \,\} \in (F)_a$.*

2. *If $|\operatorname{domain}(a_0)| = |\operatorname{domain}(a_1)| = \lambda$ and $a_1 \supseteq a_0$ then $\{\, (a_0', a_1') \mid f_{a_0,a_0'} = f_{a_1,a_1'} \restriction \operatorname{domain}(a_0) \,\} \in (F)_{(a_0,a_1)}$.*

Proof. For the first clause, note that the elementarity of i^F implies that $\{\, a' \mid R_{a'} = R_a \,\} \in (F)_a$. Let r and r' be members of R_a. To see that $\{\, (a, a') \mid f_{a,a',r} = f_{a,a',r'} \,\} \in (F)_{(a,a')}$, set $\pi_{a,r,r'} = \sigma_{a,r'}^{-1} \circ \sigma_{a,r}$ and $\pi_{a',r,r'} = \sigma_{a',r'}^{-1} \circ \sigma_{a',r}$. Then by elementarity $\{\, a' \mid \pi_{a',r,r'} = \pi_{a,r,r} \restriction |\operatorname{domain}(a')| \,\} \in (F)_a$, and if a' is any member of this set, then (letting $\lambda' = |\operatorname{domain}(a')|$ and letting $\xi \in \sigma_{a,r}[\lambda']$ be arbitrary),

$$\begin{aligned} f_{a,a',r}(\xi) = \sigma_{a',r} \circ \sigma_{a,r}^{-1}(\xi) &= (\sigma_{a',r'} \circ \pi_{a',r,r'}) \circ (\sigma_{a,r'} \circ \pi_{a,r,r'})^{-1}(\xi) \\ &= (\sigma_{a',r'} \circ \pi_{a,r,r'} \restriction \lambda') \circ (\pi_{a,r,r'}^{-1} \circ \sigma_{a,r'}^{-1})(\xi) \\ &= \sigma_{a',r'} \circ \sigma_{a',r'}^{-1}(\xi) = f_{a,a',r'}(\xi). \end{aligned}$$

This completes the proof of Clause (1) of the Proposition, and a similar argument proves Clause (2). $\qquad\square$

Definition 4.8 (The one-step extension)**.** Suppose that $w \in A_\gamma^{s,\tau}$ where $\gamma \notin \operatorname{domain}(s)$ and $\tau = \min(\operatorname{domain}(s) \backslash \gamma)$. Then $s' = \operatorname{add}(s, w)$ is the condition with $\operatorname{domain}(s') = \operatorname{domain}(s) \cup \{\gamma\}$ defined as follows:

1. $s'(\gamma) = (\bar{\kappa}_\gamma^w, \vec{F}^w, z^w \restriction [\gamma_0, \gamma], \vec{A}^w)$.
2. $s'(\tau) = (\bar{\kappa}_\tau^s, \vec{F}^{s',\tau}, z^{s',\tau}, \vec{A}^{s',\tau})$ where

---

(a) $\bar\kappa^{s'}_\tau = \bar\kappa^s_\tau$ and $\vec{F}^{s',\tau} = \vec{F}^{s,\tau}{\restriction}(\gamma,\tau)$,

(b) $z^{s',\tau}$ is obtained from $z^{s,\tau}{\restriction}(\gamma,\tau]$ by using Definition 4.6 to replace $f^{s,\tau}_{\nu\nu'}$ with $f^{s',\tau}_{\nu,\nu'} = f^{s,\tau}_{\nu,\nu'} \cup f_{a^{s,\tau}_{\nu,\nu'}, a^w_{\nu,\nu'}}$ whenever $\tau \geqslant \nu > \gamma \geqslant \nu' \geqslant \gamma_0$, and

(c) if $\gamma < \nu < \tau$ then $A^{s',\tau}_\nu = \{\, \sigma(w') \mid w' \in A^{s,\tau}_\nu \wedge \bar\kappa^w_\gamma < \bar\kappa^{w'}_\nu \,\}$, where

$$\sigma(w'){\restriction}[\gamma,\nu] = \mathrm{add}(w'{\restriction}[\gamma_0,\nu], w{\uparrow}\nu) \qquad (6)$$
$$\sigma(w'){\restriction}(\nu,\tau) = w'{\restriction}(\nu,\tau).$$

3. $s'(\gamma') = s(\gamma')$ for all $\gamma' \in \mathrm{domain}(s')\backslash\{\gamma,\tau\}$.

Note that Equation (6) uses recursion on the pair (γ,τ), along with the fact that $w'{\restriction}[\gamma_0,\nu] \in P^*_\nu$.

If any part of the definition of $\mathrm{add}(s,w)$ cannot be carried out as described, then $\mathrm{add}(s,w)$ is undefined. Note that the set of w for which it is defined is a member of $U^{s,\tau}_\gamma$, so that we can assume without loss of generality that $\mathrm{add}(s,w)$ is defined for all $w \in A^{s,\tau}_\gamma$.

This completes the definition of the one-step extension.

4.2.2 The direct extension order $\leqslant^*$.

The direct extension order $\leqslant^*$ is the restriction of the forcing order $\leqslant$ to the pairs (s',s) such that $\mathrm{domain}(s) = \mathrm{domain}(s')$. Again, the definition uses recursion on τ:

Definition 4.9. If $s',s \in P(\vec{F})$ then $s' \leqslant^* s$ if $\mathrm{domain}(s') = \mathrm{domain}(s)$ and $s'(\tau) \leqslant^* s(\tau)$ for all $\tau \in \mathrm{domain}(s)$. The ordering $s'(\tau) \leqslant^* s(\tau)$ on P^*_τ holds if and only if the following conditions hold:

1. $\bar\kappa^{s',\tau} = \bar\kappa^{s,\tau}$ and $\vec{F}^{s',\tau} = \vec{F}^{s,\tau}$.

2. $a^{s',\tau}_{\gamma,\gamma'} \supseteq a^{s,\tau}_{\gamma,\gamma'}$ for each pair (γ,γ') for which they are defined.

3. For each $\gamma \in (\gamma_0,\tau)$ and each $w' \in A^{s',\tau}_\gamma$ there is $w \in A^{s,\tau}_\gamma$ such that

   (a) $w'{\restriction}[\gamma_0,\gamma] \leqslant^* w{\restriction}[\gamma_0,\gamma]$ in P^*_γ.

   (b) $a^{w'}_{\nu,\nu'} \supseteq a^w_{\nu,\nu'}$ for $\tau \geqslant \nu > \gamma \geqslant \nu' \geqslant \gamma_0$.

   (c) For all pairs (ν,ν') with $\tau \geqslant \nu > \nu' \geqslant \gamma_0$ we have $f_{a^{s',\tau}_{\nu,\nu'}, a^{w'}_{\nu,\nu'}} \supseteq f_{a^{s,\tau}_{\nu,\nu'}, a^w_{\nu,\nu'}}$, where these two functions are as defined in Definition 4.6.

4. $f^{s',\tau}_{\nu,\nu'} \supseteq f^{s,\tau}_{\nu,\nu'}$ for each pair ν,ν' for which they are defined.

Clause 3 implies that $\mathrm{add}(s',w') \leqslant^* \mathrm{add}(s,w)$. This clause corresponds to the requirement in Prikry forcing that $A^{s'} \subseteq A^s$; however the ultrafilters $U^{s,\tau}_\gamma$ used in this forcing vary with s. Gitik [3] also has varying ultrafilters, but takes them from a predefined set and uses predefined witnesses to a Rudin-Keisler

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order on the ultrafilters. Our definition could also be stated in terms of the Rudin-Keisler order, however the ultrafilters would have to be defined on the complete Boolean algebra induced by the ordering $(P^*_{\tau,\gamma}, \leqslant^*)$.

This completes the definition of the forcing $(P(\vec{F}), \leqslant^*, \leqslant)$.

4.3 Properties of the forcing $P(\vec{F})$

Definition 4.10. If $\vec{w}$ is a sequence of length n, then we write $\mathrm{add}(s, \vec{w})$ for the condition defined by recursion as $\mathrm{add}(s, \vec{w}) = s$ if $n = 0$, and $\mathrm{add}(s, \vec{w}) = \mathrm{add}(\mathrm{add}(s, \vec{w} \restriction (n-1)), w_{n-1})$ if $n > 0$.

Proposition 4.11. *Suppose that $s \leqslant t$. Then there is $\vec{z}$ such that $s \leqslant^* \mathrm{add}(t, \vec{z}) \leqslant t$*

Proof. The proposition will follow by an easy induction on the length of $\vec{z}$ once we show that for any $t' \leqslant^* t$ and $s = \mathrm{add}(t', w') \leqslant t'$, where $w' \in A^{t',\tau}_\gamma$, there is $w \in A^{t;\tau}_\gamma$ such that $s \leqslant^* \mathrm{add}(t, w) < t$. Clause 3 of the Definition 4.9 of the direct ordering $\leqslant^*$ is designed to provide such a w:

$$
\begin{aligned}
s(\tau) &\leqslant^* \mathrm{add}(t, w)(\tau) && \text{by Clauses (3b,c),} \\
s(\gamma) &\leqslant^* \mathrm{add}(t, w)(\gamma) && \text{by Clause (3a), and} \\
s(\gamma') &= \mathrm{add}(t', w')(\gamma') = t'(\gamma') \\
&\leqslant^* t(\gamma') = \mathrm{add}(t, w)(\gamma') && \text{for } \gamma' \in \mathrm{domain}(s') \backslash \{\tau, \gamma\}.
\end{aligned}
$$

$\square$

Proposition 4.12. *Suppose $s \leqslant t$ and $\gamma \in \mathrm{domain}(s) \backslash \mathrm{domain}(t)$, and let $\tau = \min(\mathrm{domain}(t) \backslash \gamma)$. Then there is $w \in A^{t;\tau}_\gamma$ such that $s \leqslant \mathrm{add}(t, w) < t$.*

Proof. By using Proposition 4.11, we can find $\vec{w}$ so that $s \leqslant^* \mathrm{add}(t, \vec{w}) \leqslant t$ for some sequence $\vec{w}$. Thus it only remains to show that the order of the sequence $\vec{w}$ can be permuted, that is, that there is $\vec{w}'$ such that $\mathrm{add}(s, \vec{w}) = \mathrm{add}(s, \vec{w}')$ and $w'_0 \in A^{s,\tau}_\gamma$.

This will follow by an easy induction once we show that the order of two consecutive one-step extensions can be reversed. Thus suppose that $s = \mathrm{add}(\mathrm{add}(t, w_0), w_1)$, with $w_0 \in A^{t,\tau_0}_{\nu_0}$ and $w_1 \in A^{\mathrm{add}(t,w_0),\tau_1}_{\nu_1}$. We want to find $w'_1 \in A^{t,\tau'_1}_{\nu_1}$ and $w'_0 \in A^{\mathrm{add}(t,w'_1),\tau'_0}_{\nu_1}$ so that $s = \mathrm{add}(\mathrm{add}(t, w'_1), w'_0)$. We have three cases:

CASE 1 ($\nu_0 < \nu_1 < \tau_0$). In this case $\tau_1 = \tau_0$, and by definition 4.9, there is $w'_1 \in A^{t,\tau_0}_{\nu_1}$ such that $w_1 = (w'_1) \!\uparrow\! \nu_1$. Then $s = \mathrm{add}(\mathrm{add}(t, w'_1), \sigma_{\nu_0}(w_0))$, where σ_{ν_0} is as defined in Clause 3 of Definition 4.8.

CASE 2 ($\nu_1 < \tau'_1 = \nu_0$). By Definition 4.8, $w_1 = \sigma_{\nu_1}(w'_1)$ for some $w'_1 \in A^{t,\tau_0}_{\nu_1}$. Then $s = \mathrm{add}(\mathrm{add}(t, w'_1), w_1 \restriction (\nu_0, \tau_0])$.

CASE 3 ($\nu_1 > \tau_0$ or $\tau'_1 < \nu_0$). In this case $\mathrm{add}(\mathrm{add}(t, w_0), w_1) = \mathrm{add}(\mathrm{add}(t, w_1), w_0)$ so we can take $w'_0 = w_0$ and $w'_1 = w_1$. $\square$

We write $P(\vec{F})\|_s$ for $\{\, s' \in P(\vec{F}) \mid s' \leqslant s \,\}$. The proof of the following proposition is straightforward.

Proposition 4.13 (Factorization). *Suppose $s \in P(\vec{F})$ and $\gamma \in \mathrm{domain}(s)$ for some $\gamma < \zeta$. Then*

$$P(\vec{F})\|_s \text{ is a regular suborder of } P(\vec{F}^{s,\gamma})\|_{s\restriction\gamma+1} \times P' \tag{7}$$

where $P' = \{\, q\restriction(\gamma,\zeta] \mid q \leqslant s \,\}$. Thus $P(\vec{F})\|_s$ can be written in the form

$$P(\vec{F})\|_s \equiv P(\vec{F}^{s,\gamma})\|_{(s\restriction\gamma+1)} \star \dot{R} \tag{8}$$

where $\dot{R}$ is a $P(\vec{F}^{s,\gamma})\|_{s\restriction\gamma+1}$-name for a Prikry style forcing order. $\square$

This factorization property is an important property of this Magidor-Radin style of Prikry forcing. Typically, equation (7) would be an equality rather than a subalgebra; however that is not true here because of the peculiar form of the Cohen conditions $f_{\nu,\nu'}^z(\xi) = h_{\nu'',\nu'}(\xi'')$ in Clause (2b) of Definition 4.4. When $\nu > \gamma \geqslant \nu''$, the determination via Definition 4.6 of the ultimate value of $h_{\nu,\nu'}$ depends on both $P(\vec{F}^{s,\gamma})\|_{s\restriction\gamma+1}$ and R. The generic $G \subseteq P(\vec{F})$ obtained from a generic $G_0 \times G_1 \subseteq P(\vec{F}^{s,\gamma}) \times P'$ is obtained by resolving, as specified in equation (1), the values of the Cohen conditions in G_1 which have the form described in Definition 4.4(2b): that is, $f_{\nu,\nu'}(\xi) = h_{\nu'',\nu'}(\xi'')$ for some ν, ν'' and ν' with $\nu > \gamma \geqslant \nu'$.

Note that the forcing P' in equation (7) is in fact identical to $P(\vec{F})$ except that the domain of the conditions is contained in the interval $[\gamma + 1, \zeta]$ instead of $[0, \zeta]$, and $\gamma + 1$ is used instead of 0 as the default value of γ_0 in the definition of P_τ^* when $\mathrm{domain}(s) \cap \tau = \varnothing$ (but the tableau of figure 1 retains all of its columns, starting with 0). Thus all of the properties proved of $P(\vec{F})$ are also true of P'. This factorization will frequently be used in proofs, sometimes implicitly, to justify simplifying notation by proving that the result holds for the case when $\mathrm{domain}(s) = \{\zeta\}$. The result then follows for arbitrary s by a simple induction on ζ: If s is an arbitrary condition in $P(\vec{F})$ and $\gamma = \max(\mathrm{domain}(s) \cap \zeta)$ then the induction step uses the induction hypothesis for $P(\vec{F}^{s,\gamma})$ and the special case $\mathrm{domain}(s) = \{\zeta\}$ for R.

Lemma 4.14 (Closure). *Suppose that $\langle\, s_\nu \mid \nu < \beta \,\rangle$ is a $<^*$-descending sequence of conditions in $P(\vec{F})$.*

1. *(κ closure) If $\beta < \bar{\kappa}^{s_0,\min(\mathrm{domain}(s_0))}$ then the infimum $\bigwedge_{\nu<\beta} s_\nu$ of this sequence exists.*

2. *(Diagonal closure) Suppose that $\beta = \bar{\kappa}^{s_0,\min(\mathrm{domain}(s_0))}$. Then there is $s = \bigtriangleup_{\nu<\beta} s_\nu \leqslant^* s_0$ such that $s \Vdash \forall \nu < \dot{\bar{\kappa}}_0 \; s_\nu \in \dot{G}$.*

Note that for the factorization forcing P' of Proposition 4.13, $\bar{\kappa}_0$ can be replaced by $\bar{\kappa}_{\gamma+1}$.

Proof. The proof is by induction on ζ, using Proposition 4.13. Thus we can assume that domain$(s_0) = \{\zeta\}$. Since the first two coordinates of $s_\nu(\zeta)$ are fixed and the third, $z^{s,\nu}$, is κ^+-closed, the fourth coordinate, $\vec{A}^{s_\nu,\zeta}$, is the only problem.

If $w', w \in P^*_{\zeta,\eta}$ then we write $w' \leqslant^* w$ if the conditions of Definition 4.9(3) hold. If $\zeta > \gamma > \eta$ then the induction hypothesis trivially extends to sequences in $P^*_{\gamma,\eta}$, since only subclause (3a) is problematic.

Now, to prove Clause(1) of the Lemma we need to define $A^{s,\zeta}_\eta$ for each $\eta < \zeta$. We can assume that $\beta < \bar{\kappa}^w_\eta$ for all $w \in A^{s,\zeta}_\eta$. Set

$$A^{s,\zeta}_\eta = \{ \bigwedge_{\nu < \beta} w_\nu \mid (\forall \nu < \beta)\, w_\nu \in A^{s_\nu,\zeta}_\eta \wedge (\forall \nu' < \nu < \beta)\, w_{\nu'} \leqslant^* w_\nu \}.$$

To see that $A^{s,\zeta}_\eta \in U^{s,\zeta}_\eta$ note that the induction hypothesis implies that the infimum $w = \bigwedge_{\nu < \beta}(i^{F^{s,\zeta}_\eta}(s_\nu)){\uparrow}\eta$ exists, and $w \in i^{F^{s,\zeta}_\eta}(A^{s,\zeta}_\eta)$.

This concludes the proof of Clause (1), and the proof of Clause (2) is similar.
$\square$

Lemma 4.15. *Suppose that $s \in P(\vec{F})$ and for all $w \in A^{s,\zeta}_\gamma$ the set D is open and dense in $(P(\vec{F}), \leqslant^*)$ below* add(s, w)*. Then there is a condition $s' \leqslant^* s$ such that $s'' \in D$ for all $s'' < s'$ having $\gamma \in$* domain(s'')*.*

Proof. By Proposition 4.12 it will be enough to show that there is $s' \leqslant^* s$ such that add$(s', w) \in D$ for all $w \in A^{s',\zeta}_\gamma$. In order to simplify notation, we assume that domain$(s) = \{\zeta\}$.

By proposition 1.6 we can assume that $A^{s,\zeta}_\gamma$ can be enumerated as $\{ w_\nu \mid \nu < \kappa \}$ so that $\nu' \leqslant \nu$ implies $\bar{\kappa}^{w_{\nu'}}_\gamma \leqslant \bar{\kappa}^{w_\nu}_\gamma$. We will define by recursion on ν a $\leqslant^*$-decreasing sequence of conditions $\langle s_\nu \mid \nu < \kappa \rangle$ in R so that add$(s_\nu, w_\nu) \leqslant^*$ add(s, w_ν) for all $\nu < \kappa$. At the same time we will define a function $\sigma\colon A^{s,\zeta}_\gamma \to P^*_{\zeta,\gamma}$ so that s_ν and $\sigma(w_\nu)$ satisfy the following conditions:

1. $s_0 = s$,

2. $s_\nu{\uparrow}\gamma = s{\uparrow}\gamma$ and $\vec{A}^{s_\nu,\zeta}{\restriction}\gamma + 1 = \vec{A}^{s,\zeta}{\restriction}\gamma + 1$ for all $\nu < \kappa$,

3. add$(s_{\nu+1}, \sigma(w_\nu)) \in D$, and

4. $s_{\nu'} \leqslant^* s_\nu$ for all $\nu' < \nu < \kappa$.

Note that clause (2) implies that add(s_ν, w) exists for all $\nu < \kappa$ and all $w \in A^{s,\zeta}_\gamma$. Also, clauses (2) and (4) imply that add$(s_{\nu'}, w_\nu) \leqslant^*$ add$(s_\nu, w_\nu) \leqslant^*$ add(s, w_ν) for all $\nu < \nu' < \kappa$.

To define the sequence, set $s_0 = s$, and if ν is a limit ordinal then set $s_\nu = \bigwedge_{\nu' < \nu} s_{\nu'}$. For a successor ordinal $\nu + 1$, since add$(s_\nu, w_\nu) \leqslant^*$ add(s, w_ν), the hypothesis implies that there is $t \leqslant^*$ add(s_ν, w_ν) such that $t \in D$.

Define $\sigma(w_\nu)$ by

$$\sigma(w_\nu){\restriction}[\gamma_0, \gamma] = (t{\uparrow}\gamma){\restriction}[\gamma_0, \gamma], \text{ and}$$
$$\sigma(w_\nu){\restriction}(\gamma, \zeta] = w_\nu{\restriction}(\gamma, \zeta].$$

By clause (2) we have $s_{\nu+1}\!\restriction\!\gamma = s\!\restriction\!\gamma$ and $\vec{A}^{s_{\nu+1},\zeta}\!\restriction\!\gamma + 1 = \vec{A}^s\!\restriction\!\gamma + 1$. The remainder of $z^{s_{\nu+1}}$ is taken from t; that is:

$$
\begin{aligned}
a^{s_{\nu+1},\zeta}_{\eta,\eta'} &= a^{t,\zeta}_{\eta,\eta'} && \text{if } \eta' > \gamma, \\
f^{s_{\nu+1},\zeta}_{\eta,\eta'} &= f^{t,\zeta}_{\eta,\eta'} && \text{if } \eta' > \gamma, \text{ and} \\
f^{s_{\nu+1},\zeta}_{\eta,\eta'} &= f^{t,\zeta}_{\eta,\eta'}\!\restriction\!(\kappa^+ \backslash \operatorname{domain}(a^{s,\zeta}_{\eta,\eta'})) && \text{if } \eta > \gamma \geqslant \gamma'.
\end{aligned}
$$

The definition of $A^{s_{\nu+1},\zeta}_\eta$ for $\zeta > \eta > \gamma$ is by recursion on γ. For $w \in A^{s_\nu,\zeta}_\eta$ and $w' \in A^{t,\zeta}_\eta$, let us write $w' \leqslant^* w$ if they satisfy Definition 4.9(3), in which case let $\pi_{w'}(w)$ be given by

1. $\pi_{w'}(w)\!\restriction\![\gamma + 1, \zeta] = w\!\restriction\![\gamma + 1, \zeta]$, and

2. $\pi_{w'}(w)\!\restriction\![\gamma_0, \gamma]$ is defined in the same way as $s_{\nu+1}$, but with $w'\!\restriction\![\gamma_0, \gamma]$, $w\!\restriction\![\gamma_0, \gamma]$ and η in place of t, s_ν and ζ.

Then

$$
A^{s_{\nu+1},\zeta}_\eta = \{\, \pi_{w'}(w) \mid w' \in A^t_\eta \wedge \bar{\kappa}^{w'} > \bar{\kappa}^{w_\nu} \wedge w \in A^{s_\nu}_\eta \wedge w' \leqslant^* w \,\}.
$$

Now set $s_\kappa = \bigwedge_{\nu < \kappa} s_\nu$, and set $\bar{w} = [\sigma]_{U^{s,\zeta}_\gamma} = i^{F^{s,\zeta}_\gamma}(\sigma)(s\!\uparrow\!\gamma)$. Then clause (2) of the initial conditions on s_ν allow $\bar{w}$ to be merged into s_κ, giving the desired extension $s' \leqslant^* s$. We can assume without loss of generality that $w' \in A^{s_\kappa,\zeta}_\eta$ whenever $w \in A^{s_\kappa,\zeta}_\eta$ and $w' \leqslant^* w$ in the sense of Definition 4.9(3).

$$
\begin{aligned}
A^{s',\zeta}_\eta &= \begin{cases} A^{\bar{w}}_\eta & \text{if } \eta < \gamma, \\ \{\, \sigma(w) \mid w \in A^s_\gamma \,\} & \text{if } \eta = \gamma \text{ and} \\ A^{s_\kappa,\zeta}_\eta & \text{if } \eta > \gamma; \end{cases} \\
z^{s',\zeta}\!\uparrow\!\gamma &= \bar{w}\!\restriction\![\gamma_0, \gamma], \\
f^{s',\zeta}_{\eta,\eta'} &= f^{s_\kappa,\zeta}_{\eta,\eta'} && \text{if } \eta > \gamma, \text{ and} \\
a^{s',\zeta}_{\eta,\eta'} &= a^{s_\kappa,\zeta}_{\eta,\eta'} && \text{if } \eta' > \gamma.
\end{aligned}
$$

$\square$

4.4 The Prikry property

Lemma 4.16. *1. Let φ be a sentence and s a condition in $P(\vec{F})$. Then there is an $s' \leqslant^* s$ such that s' decides φ.*

2. Let D be a dense subset of $P(\vec{F})$, and suppose $s \in P(\vec{F})$. Then there is an $s' \leqslant^* s$ and a finite $b \subseteq \zeta + 1$ such that any $s'' \leqslant s'$ with $b \subseteq \operatorname{domain}(s'')$ is a member of D.

Proof of Lemma 4.16. The proof of Lemma 4.16 is by induction on the length ζ of $\vec{F}$. By the induction hypothesis and Proposition 4.13 we can simplify the notation by assuming that $\mathrm{domain}(s) = \{\zeta\}$. The main part of the proof is the following claim:

Claim 4.16.1. *Suppose that $D \subseteq P(\vec{F})$ is dense and $s \in P(\vec{F})$ has domain $\{\zeta\}$. Then there is $s' \leqslant^* s$ such that either $s' \in D$ or for some $\gamma < \zeta$*

$$s' \Vdash (\exists w \in A_\gamma^{s',\zeta})(\exists t \in \dot{G} \cap D)\left(\mathrm{domain}(t) \subseteq (\gamma+1) \cup \{\zeta\}\right.$$
$$\left. \wedge\, t \leqslant \mathrm{add}(s',w) \wedge t(\zeta) = \mathrm{add}(s',w)(\zeta)\right) \quad (9)$$

Proof. For each $\gamma < \zeta$, define

$$D_\gamma^+ = \{\, t \in P(\vec{F}) \mid t \Vdash (\exists t' \in \dot{G} \cap D)\ \mathrm{domain}(t') \subseteq (\gamma+1) \cup \{\zeta\}\,\}$$
$$D_\gamma^- = \{\, t \in P(\vec{F}) \mid t \Vdash \neg(\exists t' \in \dot{G} \cap D)\ \mathrm{domain}(t') \subseteq (\gamma+1) \cup \{\zeta\}\,\}$$
$$E_\gamma = \{\, t \in P(\vec{F}) \mid (\forall t' \leqslant t)\left(t' \in D \wedge \mathrm{domain}(t') \subseteq (\gamma+1) \cup \{\zeta\}\right.$$
$$\left. \implies (t'{\upharpoonright}(\gamma+1) \cup t{\upharpoonright}\{\zeta\}) \in D\right)\}.$$

First, suppose that for all $\gamma < \zeta$ the set $(D_\gamma^+ \cup D_\gamma^-) \cap E_\gamma$ is $\leqslant^*$-dense below any condition $t \leqslant s$ with $\mathrm{domain}(t) = \{\gamma,\zeta\}$. Then by Lemma 4.15 there is $s' \leqslant^* s$ such that for each $\gamma < \zeta$ and $w \in A_\gamma^{s',\zeta}$ we have $\mathrm{add}(s,w) \in (D_\gamma^+ \cup D_\gamma^-) \cap E_\gamma$. By shrinking the sets $A_\gamma^{s',\zeta}$ we can assume that for each γ, $\{\, \mathrm{add}(s',w) \mid w \in A_\gamma^{s',\zeta}\,\}$ is contained in one of $D_\gamma^+ \cap E_\gamma$ or $D_\gamma^- \cap E_\gamma$. Since D is dense it follows that $\{\, \mathrm{add}(s',w) \mid w \in A_\gamma^{s',\zeta}\,\} \subseteq D_\gamma^+$ for some $\gamma < \zeta$, and it follows by Proposition 4.13 that s' satisfies the formula (9).

Now fix $\gamma < \zeta$ and $t \leqslant s$ with $\gamma \in \mathrm{domain}(t)$. We will show that $(D_\gamma^+ \cup D_\gamma^-) \cap E_\gamma$ is $\leqslant^*$-dense below t. First, note that by Proposition 4.13, the set E_γ is $\leqslant^*$-dense below any condition t with $\gamma \in \mathrm{domain}(t)$. Now for $t \in E_\gamma$, consider the following formula in the forcing language of $P(\vec{F}^{t,\gamma})$:

$$\exists t' \in \dot{G}\,(t' \cup t{\upharpoonright}\{\gamma\} \in D). \quad (10)$$

By the induction hypothesis of Lemma 4.16(1) there is $t'' \leqslant^* t{\upharpoonright}\gamma + 1$ which decides, in $P(\vec{F}^{t,\eta})$, the truth of formula (10). Then $t'' \cup t{\upharpoonright}\{\zeta\}$ is in either D_γ^+ or in D_γ^-. $\qquad\square$

To complete the proof of Lemma 4.16(1), apply Claim 4.16.1 with $D = \{\, t \mid t \parallel \varphi\,\}$. Since we are done if there is $s' \leqslant^* s$ in D we can assume by Claim 4.16.1 that there is $s' \leqslant^* s$ and $\gamma < \zeta$ such that (9) holds.

By the induction hypothesis, for each $w \in A_\gamma^{s',\zeta}$ there is $t_w \leqslant^* \mathrm{add}(s',w){\upharpoonright}(\gamma+1)$ in $P(\vec{F}^w)$ such that $t_w \cup \mathrm{add}(s',w){\upharpoonright}\{\zeta\} \parallel \varphi$. Then either $\{\, w \in A_\gamma^{s',\zeta} \mid t_w \cup \mathrm{add}(s',w){\upharpoonright}\{\gamma\} \Vdash \varphi\,\} \in U_\gamma^{s',\zeta}$ or $\{\, w \in A_\gamma^{s',\zeta} \mid t_w \cup \mathrm{add}(s',w){\upharpoonright}\{\gamma\} \Vdash \neg\varphi\,\} \in U_\gamma^{s',\zeta}$. Now reduce $A_\gamma^{s',\zeta}$ to whichever set is in $U_\gamma^{s',\zeta}$, and apply Lemma 4.15 to obtain $s'' \leqslant^* s'$ such that s'' decides φ.

Lemma 4.16(2) is proved similarly, applying Claim 4.16.1 using the set D given in the hypothesis. □

If $\gamma < \zeta$ and $G \subseteq P(\vec{F})$ is generic, then set $G{\upharpoonright}\gamma + 1 = \{\, s{\upharpoonright}\gamma + 1 \mid \gamma \in \operatorname{domain}(s) \wedge s \in G \,\}$. Then $G{\upharpoonright}\gamma + 1$ is a generic subset of $P(\vec{F}^{s,\gamma})$.

Corollary 4.17 (No new bounded sets). *Suppose $x \in M[G] \backslash M$ and $x \subset \lambda < \bar{\kappa}_{\gamma+1}^{M[G]}$. Then $x \in M[G{\upharpoonright}\gamma' + 1]$ for some $\gamma' < \gamma$.*

Proof. If $\gamma = \gamma' + 1$ then Propositions 4.13 and 4.14 imply that γ' is as required. If γ is a limit ordinal then take γ' least such that $\bar{\kappa}_{\gamma'}^G > \lambda$. □

Corollary 4.18. *If $\vec{F}$ is a suitable sequence with critical point κ then $P(\vec{F})$ has the κ-approximation property: if $G \subseteq P(\vec{F})$ is M-generic then for any function $f \in M[G]$ with $\operatorname{domain}(f) = \kappa$ there is a set $A \in M$ with $|A| \leqslant \kappa$ and $\operatorname{range}(f) \subseteq A$.*

Proof. Let $\dot{f}$ be the name of a function $f \colon \kappa = \bar{\kappa}_\zeta \to \kappa^+$, and let s be a condition, which we will assume has domain $\{\zeta\}$.

If $\zeta = \gamma + 1$ then, for any condition s with $\gamma \in \operatorname{domain}(s)$, factor $P(\vec{F})\|_s$ as $P(\vec{F}^{s,\gamma})\|_{s{\upharpoonright}\gamma+1} \times P'$. Then P' is κ^+-closed since $\vec{F}^{s,\gamma+1} = \varnothing$, so there is $s' \leqslant s{\upharpoonright}\{\zeta\}$ such that for all $\alpha < \kappa$ there are β and $t \in G{\upharpoonright}\gamma + 1$ such that $t \cup s' \Vdash \dot{f}(\alpha) = \beta$. Thus we can take

$$A = \{\, \beta \mid (\exists \alpha < \kappa)(\exists t \in P(\vec{F}^{s,\gamma})\|_{s{\upharpoonright}\gamma+1})\, t \cup s' \Vdash \dot{f}(\alpha) = \beta \,\}.$$

If ζ is a limit ordinal then use Lemma 4.14 to define a $\leqslant^*$-decreasing sequence of conditions $s_\gamma \leqslant^* s$ such that s_γ forces the following formula:

$$(\forall \alpha < \dot{\bar{\kappa}}_\gamma)\forall \beta \, \big((\exists t \in \dot{G})\ \operatorname{domain}(t) \subseteq (\gamma + 1 \cup \{\zeta\}) \wedge t \Vdash \dot{f}(\alpha) = \beta \implies$$
$$(\exists w \in A_\gamma^{s_\gamma, \zeta})(\exists t \in \dot{G}{\upharpoonright}\gamma + 1)\, (t \leqslant \operatorname{add}(s_\gamma, w) \wedge$$
$$t \cup \operatorname{add}(s_\gamma, w){\upharpoonright}\{\zeta\} \Vdash \dot{f}(\alpha) = \beta \big).$$

Set $s' = \bigwedge_{\nu < \zeta} s_\nu$ and

$$A_\gamma = \{\, \beta \mid \exists \alpha (\exists w \in A_\gamma^{s', \zeta})(\exists t < \operatorname{add}(s', w))$$
$$t{\upharpoonright}(\gamma, \zeta] = \operatorname{add}(s, w){\upharpoonright}(\gamma, \zeta] \wedge t \Vdash \dot{f}(\alpha) = \beta \,\}.$$

Then $s' \Vdash \operatorname{range}(\dot{f}) \subseteq \bigcup_{\gamma < \zeta} A_\gamma$. □

Corollary 4.19. *Forcing with $P(\vec{F})$ does not collapse any cardinal which is not in the set $\bigcup_{\gamma \leqslant \zeta}[\bar{\kappa}_\gamma^{++}, \bar{\kappa}_\gamma^{+\gamma+1}]$.*

Proof. Suppose λ is a cardinal of M which is collapsed in $M[G]$, where $G \subseteq P(\vec{F})$ is M-generic. If $\lambda < \kappa = \bar{\kappa}_\zeta$ then Corollary 4.17 implies that the collapsing function is in $M[G{\upharpoonright}\gamma + 1]$ for some $\gamma < \zeta$. Thus we can assume without loss of generality that $\gamma = \zeta$ and $\lambda \geqslant \kappa$. Also $\lambda \leqslant |P(\vec{f})| \leqslant \kappa^{+(\zeta+1)}$. Finally, Lemma 4.18 implies that $\lambda \neq \kappa^+$. □

In the forcing of Gitik from which this forcing is derived, a preliminary forcing is used to define a morass-like structure which guides the main forcing so that no cardinals are collapsed. We omit this preliminary forcing as unnecessary for the proof of the main theorem; however as a consequence we do not know whether the cardinals of M_Ω which are excepted in Lemma 4.19 are cardinals in the Chang model.

4.5 Introducing the equivalence relation

We now proceed to the second part of the definition of the forcing by adding a variant of Gitik's equivalence relation $\leftrightarrow$ on $P(\vec{F})$. Recall that if F is an extender on λ then $(F)_b$ is the ultrafilter $\{\, x \in V_\lambda \mid b \in i^F(x) \,\}$.

Definition 4.20. Suppose that $\vec{F}$ is a suitable sequence of extenders of length at least $\gamma + 1$ on a cardinal λ, and $a, a' \colon x \to \mathrm{supp}(F_\gamma)$ for some $x \subseteq [\lambda, \lambda^+)$ of size λ. Set $Y = \bigcup_{\gamma < \gamma'} \mathrm{supp}(F'_\gamma)$.

1. $a \leftrightarrow_0 a'$ if $(F_\gamma)_{y \cup \{a\}} = (F_\gamma)_{y \cup \{a'\}}$ for all $y \in [Y]^{<\omega}$.

2. If $n \geq 0$ then $a \leftrightarrow_{n+1} a'$ if for all $b \supseteq a$ there is $b' \supseteq a'$ such that $b \leftrightarrow_n b'$, and for all $b' \supseteq a'$ there is $b \supseteq a$ such that $b \leftrightarrow_n b'$.

Definition 4.21. We write $\mathcal{N}$ for the set of sequences $\vec{n} \in {}^\zeta\omega$ such that $\{\, \iota < \zeta \mid n_\iota < m \,\}$ is finite for each $m \in \omega$. Suppose that $\vec{F}$ is a suitable sequence of extenders on λ and $\vec{a}$ and $\vec{a}'$ are sequences with $\mathrm{domain}(\vec{a}) = \mathrm{domain}(\vec{a}') = \mathrm{domain}(\vec{F}) \subseteq \zeta$.

1. If $\vec{n} \in \mathcal{N}$ then $\vec{a} \leftrightarrow_{\vec{n}} \vec{a}'$ if $a_\nu \leftrightarrow_{n_\nu} a'_\nu$ for all $\nu \in \mathrm{domain}(\vec{F})$.

2. $\vec{a} \leftrightarrow \vec{a}'$ if there is some $\vec{n} \in \mathcal{N}$ such that $\vec{a} \leftrightarrow_{\vec{n}} \vec{a}'$.

Definition 4.22. The extension of $\leftrightarrow_{\vec{n}}$ to P^*_γ is by recursion on γ: we assume that its restriction to P^*_η is defined for all $\eta < \gamma$.

If $\eta < \gamma$ and $w, w' \in P^*_{\gamma,\eta}$ then $w \leftrightarrow_{\vec{n}} w'$ if (i) $w \restriction [\gamma_0, \eta] \leftrightarrow_{\vec{n}} w' \restriction [\gamma_0, \eta]$, as members of P^*_η, and (ii) $w \restriction [\eta+1, \gamma) = w' \restriction [\eta+1, \gamma)$.

Suppose $t, t' \in P^*_\gamma$. Then $t \leftrightarrow_{\vec{n}} t'$ if the following conditions hold:

1. $\bar{\kappa}^t = \bar{\kappa}^{t'}$ and $\vec{F}^t = \vec{F}^{t'}$.

2. $f^t_{\nu,\nu'} = f^{t'}_{\nu,\nu'}$ for all ν, ν' for which they are defined.

3. $a^t_{\mu,\nu} \leftrightarrow_{n_\nu} a^{t'}_{\mu,\nu}$ for all $\gamma \geq \mu > \nu$.

4. $[A^t_\nu]_{\vec{n}} = [A^{t'}_\nu]_{\vec{n}}$ for all $\nu \in \mathrm{domain}(\vec{F}^t)$, where $[A]_{\vec{n}} = \{\, [w]_{\leftrightarrow_{\vec{n}}} \mid w \in A \,\}$.

Finally, $s \leftrightarrow_{\vec{n}} s'$ for conditions $s, s' \in P(\vec{F})$ if $\mathrm{domain}(s) = \mathrm{domain}(s')$ and $s(\gamma) \leftrightarrow_{\vec{n}} s'(\gamma)$ for all γ in their common domain.

It is easy to see that $\leftrightarrow$ is an equivalence relation.

Proposition 4.23. *Suppose that* $\mathrm{add}(s, \vec{z}) \leqslant s \leftrightarrow_{\vec{n}} t$. *Then there is* $\vec{w}$ *such that* $\mathrm{add}(s, \vec{z}) \leftrightarrow_{\vec{n}} \mathrm{add}(t, \vec{w}) \leqslant t$.

Proof. We show that this is true when $\vec{z}$ has length one. An induction will then show that it is true in general. .

Suppose that $\mathrm{add}(s, z) \leqslant s \leftrightarrow_{\vec{n}} t$, with $z \in A_\gamma^{s, \tau}$. By definition 4.22(4) there is $w \in A_\gamma^{t, \tau}$ such that $z \leftrightarrow_{\vec{n}} w$. Then the condition $z \restriction [\gamma_0, \gamma] \leftrightarrow_{\vec{n}} w \restriction [\gamma_0, \gamma]$ implies that $\mathrm{add}(s, z)(\gamma) \leftrightarrow_{\vec{n}} \mathrm{add}(t, w)(\gamma)$, and the condition that $z \restriction [\gamma + 1, \tau) = w \restriction [\gamma + 1, \tau)$ implies that the Cohen functions induced in $\mathrm{add}(s, z)(\tau)$ and $\mathrm{add}(t, w)(\tau)$ by Definition 4.6 are equal. Therefore $\mathrm{add}(s, z)(\tau) \leftrightarrow_{\vec{n}} \mathrm{add}(t, w)(\tau)$.

Since these are the only values of s and t which are changed in the extensions, it follows that $\mathrm{add}(s, z) \leftrightarrow_{\vec{n}} \mathrm{add}(t, w)$. $\qquad\square$

Proposition 4.24. *Suppose* $s' \leqslant^* s \leftrightarrow_{\vec{n}} t$, *and that* $n_\nu > 0$ *for all* $\nu \notin$ domain(s). *Then there is* $t' \leqslant^* t$ *such that* $s' \leftrightarrow_{\vec{m}} t'$ *for all* $\nu < \zeta$, *where* $m_\nu = n_\nu - 1$ *if* $n_\nu > 0$, *and* $m_\nu = 0$ *otherwise,*

Proof. We will show by induction on γ that, under the hypotheses of the Proposition, if $\gamma \in$ domain$(s) =$ domain(t) then there is $t'(\gamma)$ such that $t'(\gamma) \leqslant^* t(\gamma)$ and $t'(\gamma) \leftrightarrow_{m_\nu} s'(\gamma)$. By the definition of $\leftrightarrow$, the sequence $\vec{F}^{t', \gamma}$ and the functions $f_{\nu, \nu'}^{t', \gamma}$ must be the same as $\vec{F}^{s', \gamma}$ and $f_{\nu, \nu'}^{s', \gamma}$. This leaves the functions $a_{\nu', \nu}^{t', \gamma}$ and sets $A_\nu^{t'}$ to be defined.

To define $\vec{a}^{t', \gamma}$, pick for each ν in the interval $\gamma_0 \leqslant \nu < \gamma$ some $b \supseteq a_{\nu+1, \nu}^{t, \gamma}$ such that $a_{\nu+1, \nu}^{s', \gamma} \leftrightarrow_{m_\nu} b$. This is possible by the definition of $\leftrightarrow_{m_\nu+1}$, since $n_\nu \neq 0$. Now set $a_{\nu+1, \nu}^{t', \gamma} = b$. By clause (4) of the Definition 4.4 of the tableau, this determines $a_{\mu, \nu}^{t', \gamma} = a_{\nu+1, \nu}^{t', \gamma} \restriction$ domain$(a_{\mu, \nu}^{s', \gamma})$ for $\mu > \nu + 1$.

Finally, set $A_\nu^{t', \gamma}$ equal to the set of all w' such that $w' \leqslant^* w$ for for some $w \in A_\nu^{t, \gamma}$ and $w' \leftrightarrow_{\vec{m}} v'$ for some $v' \in A_\nu^{s', \gamma}$. Then $[A_\nu^{t', \gamma}] = [A_\nu^{s', \gamma}]$ since for all $v' \in A_\nu^{s', \gamma}$ there is $v \in A_\nu^{s, \gamma}$ and $w \in A_\nu^{t, \gamma}$ such that $v' \leqslant^* v \leftrightarrow_{\vec{m}} w$, and then the induction hypothesis implies that there is $w' \leqslant^* w$ with $w' \leftrightarrow_{\vec{m}} v'$. $\qquad\square$

Definition 4.25. We will write $[s]$ for $[s]_\leftrightarrow = \{\, t \mid s \leftrightarrow t \,\}$. The ordering on $P(\vec{F})/\leftrightarrow$ is the smallest transitive relation such that $[s] \leqslant [t]$ holds if either $s \leqslant t$ or $s \leftrightarrow t$.

Proposition 4.26. *Suppose* $[t] = [s]$ *and* $t' \leqslant t$. *Then there are* $s'' \leqslant s$ *and* $t'' \leqslant t'$ *such that* $[s''] = [t'']$.

Proof. Suppose that $t \leftrightarrow_{\vec{n}} s$. By using a further extension $t'' = \mathrm{add}(t', \vec{w})$ we can arrange that $\{\, \nu \mid n_\nu = 0 \,\} \subseteq$ domain(t''). By Proposition 4.11 there is $\vec{z}$ so that $t'' \leqslant^* \mathrm{add}(t, \vec{z}) \leqslant t$. By Proposition 4.23 it follows that there is $\vec{w}$ so that $\mathrm{add}(t, \vec{z}) \leftrightarrow_{\vec{n}} \mathrm{add}(s, \vec{w}) \leqslant s$. Finally it follows by Proposition 4.24 that there is $s'' \leqslant^* \mathrm{add}(s, \vec{w})$ so that $s'' \leftrightarrow t''$. $\qquad\square$

Proposition 4.27. *Suppose that* $[t] \leqslant [s]$. *Then there is a condition* $q \leqslant s$ *such that* $[q] \leqslant [t]$.

Proof. If $[t] \leqslant [s]$ then there is a sequence $t = t_0 \lesssim t_1 \lesssim \cdots \lesssim t_{k-1} \lesssim t_k = s$, where we write $s \lesssim s'$ to mean that either $s \leqslant s'$ or $s \leftrightarrow s'$. We prove the proposition by induction on the length of the shortest such sequence, assuming as an induction hypothesis that there is $\bar{q} \leqslant t_{k-1}$ such that $[\bar{q}] \leqslant [t]$.

If $t_{k-1} \leqslant s$, then it follows that $\bar{q} \leqslant s$ and we can take $q = \bar{q}$. Otherwise $\bar{q} \leqslant t_{k-1} \leftrightarrow s$, and Proposition 4.26 asserts that there is $q \leqslant s$ and $q' \leqslant \bar{q}$ such that $q \leftrightarrow q'$. But then $[q] = [q'] \leqslant [t]$, as required. □

Corollary 4.28. $P(\vec{F})$ *is forcing equivalent to* $(P(\vec{F})/{\leftrightarrow}) * \dot{R}$ *where* $\dot{R}$ *is a* $P(\vec{F})/{\leftrightarrow}$*-name for a partial order.* □

Corollary 4.29. *Forcing with* $P(\vec{F})/{\leftrightarrow}$ *does not collapse any cardinal which is not in the set* $\bigcup_{\gamma \leqslant \zeta}[\bar{\kappa}_\gamma^{++}, \bar{\kappa}_\gamma^{+\omega_1})$.

Proof. By Corollary 4.19 this is true in the extension by $P(\vec{F}) = (P(\vec{F})/{\leftrightarrow}) * \dot{R}$; hence it is certainly true in the extension by $P(\vec{F})/{\leftrightarrow}$. □

4.6 Constructing a generic set

Much of the argument in this subsection is basically the same as Carmi Merimovich's first genericity argument [9, Theorem 5.1]. In order to construct a M_B-generic set we need to move outside of M_B: we work in $V[h]$, where h is a generic collapse of $\mathcal{R}$ onto ω_1 so that $|M[h]| = \omega_1$. Since this Levy collapse does not add countable sequences of ordinals the Chang model is unchanged, the ordering $\leqslant^*$ of $P(\vec{N}{\upharpoonright}\zeta)$ is still countably complete, and M is still closed under countable sequences. Furthermore, since h is generic over M, $M[h] \supseteq M(\mathcal{R})$ and $M[h]$ is mouse over h which has all of the required properties of M.

Lemma 4.30 (Generic set construction)**.** *Let* h *be a generic collapse of* $\mathcal{R}$ *onto* ω_1 *with countable conditions, and let* B *be a countable subset of* I *with* $\mathrm{otp}(B) = \zeta$. *Then there is, in* $V[h]$, *an* $i_\Omega(M_B)$*-generic set* $G \subseteq i_\Omega(P(\vec{E}{\upharpoonright}\zeta)/{\leftrightarrow})$ *such that every countable subset of* M_B *is contained in* $M_B[G]$.

4.6.1 Proof of Lemma 4.30

Since $M_B \cong M_{B(\zeta)}$, where $B(\zeta) = \{\kappa_\nu \mid \nu < \zeta\}$, containing the first ζ members of I, it will be sufficient to prove this for the case where $B = B(\zeta)$. This will simplify notation, since then $M_B|\Omega$ is transitive and $\bar{\kappa}_\nu^G$ is equal to both the νth member κ_ν of I and the νth member of B.

We define a partial order R. Our assumptions on M are sufficiently generous that the definition of R can be made inside M, using $\langle N_\xi \cap H_\tau^M \mid \xi < \omega_1 \rangle$, for some sufficiently large cardinal τ of M, instead of $\langle N_\xi \mid \xi < \omega_1 \rangle$.

Definition 4.31. $R = \bigcup_{\xi < \omega_1} R_\xi$, where R_ξ is defined as follows: The members of R_ξ are the pairs $([s], b)$ such that $[s] \in P(\vec{E}{\upharpoonright}\delta)/{\leftrightarrow}$ is a condition with $\mathrm{domain}(s) = \{\zeta\}$ and $b = \langle b_\gamma : \gamma < \zeta \rangle$ where each b_γ is a function in N_ξ satisfying the following three conditions:

1. $\mathrm{domain}(b_\gamma) = \mathrm{domain}(a^{s,\zeta}_{\gamma+1,\gamma})$ for each $\gamma < \zeta$,

2. $\mathrm{range}(b_\gamma) \subset [\kappa, \kappa^{+\omega_1})$ for each $\gamma < \zeta$, and

3. $\langle a^{s,\zeta}_{\gamma+1,\gamma} \mid \gamma < \zeta \rangle \leftrightarrow b_\gamma$.

The ordering of R is $(s', b') \leqslant (s, b)$ if $[s'] \leqslant [s]$ in $P(\vec{N})/\!\leftrightarrow$ and $(\forall \gamma < \zeta)\, b'_\gamma \supseteq b_\gamma$.

  Clause (3) requires some explanation, since $\mathrm{range}(b_\gamma) \not\subseteq \mathrm{supp}(E_\gamma) = \mathrm{supp}(E) \cap N_\gamma$. The Definition 4.20 of the relation $a \leftrightarrow_n a'$ uses the parameter γ in two ways. The first use is in the definition of $a \leftrightarrow_0 a'$, where the set $Y = \bigcup_{\gamma' < \gamma} \mathrm{supp}(F_{\gamma'})$ is used as the set of y in the requirement $(F_\gamma)_{y \cup \{a\}} = (F_\gamma)_{y \cup \{a'\}}$. Here the same set Y is used, and since $(E_\gamma)_{y \cup \{a\}} = (E)_{y \cup \{a\}}$ the requirement can be altered to $(E_\gamma)_{y \cup \{a\}} = (E)_{y \cup \{b\}}$.

  The second way in which the parameter γ is used is in the domain of the quantifiers. In Clause (3) the extensions $a' \supseteq a^{s,\zeta}_{\gamma+1,\gamma}$ are in M_γ, while the extension $b' \supseteq b_\gamma$ are in M. We reconcile these demands by using the elementarity of N_γ, and this requires expressing Clause (3) as a first order statement. This is achieved by the following Proposition, which is the reason for the requirement in Definition 4.1 that $|\bigcup_{\gamma' < \gamma} N_{\gamma'}|^{++} \subseteq N_\gamma$.

Proposition 4.32. *For any* $b\colon x \to [\kappa, \kappa^{+\omega_1})$ *with* $x \in [\kappa^+ \backslash \kappa]^\kappa$, *there is a formula* $\varphi(n, a)$, *with parameters from* N_γ, *such that if* $a\colon x \to \mathrm{supp}(E_\gamma)$ *then* $a \leftrightarrow_n b$ *if and only if* $N_\gamma \models \varphi(n, a)$.

Proof. For $n = 0$, note that the sequence of ultrafilters $\langle (E)_{y \cup \{b\}} \mid y \in [Y]^{<\omega} \rangle$ can be coded as a subset of $[Y]^{<\omega} \times \mathcal{P}(\kappa)$, which has cardinality $|Y| = |\bigcup_{\gamma' < \gamma} N_{\gamma'}|$.

  Working in M, define T to be the tree of finite sequences of the form $\langle [b_i]_{\leftrightarrow_0} \mid i < k \rangle$ where $\langle b_i \mid i < k \rangle$ is a $\subseteq$-increasing sequence of functions $b_i\colon x_i \to [\kappa, \kappa^{+\omega_1})$ with $x_i \in [\kappa^+ \backslash \kappa]^\kappa$. Since T is at most $|\mathcal{P}(Y)|$-branching, it has cardinality at most $|\mathcal{P}^2(Y)|$, so Clauses (5) and (6) of Definition 4.1 ensure that $T \in N_\gamma$.

  Write T_b for the portion of T above $\langle [b]_{\leftrightarrow_0} \rangle$. Then the conclusion of the proposition is satisfied by the formula $\varphi(n, a)$, with parameter T_b, which asserts that the first n levels of T_b and $(T_a)^{N_\gamma}$ are equal. Since $[\mathrm{supp}(E_\gamma)]^\kappa \cap N_\gamma \in N_\gamma$, this is a first order formula over N_γ. $\qquad \square$

Lemma 4.33.   1. $\{ ([s], b) \mid s \in D \}$ *is dense in* R *for each* $\leqslant^*$-*dense set* $D \subseteq P(\vec{E}{\restriction}\zeta)$ *in* M.

  2. *Suppose* $\gamma < \zeta$ *and* $\beta \in [\kappa, \kappa^{+\omega_1})$. *Then there is a dense subset of conditions* $([s], b) \in R$ *such that* $b(\xi) = \beta$ *for some* $\xi \in \mathrm{domain}(a^{s,\zeta}_{\zeta,\gamma})$.

Proof. For clause (1), let $([s], b) \in R$ be arbitrary and set $\vec{a} = \langle a^{s,\zeta}_{\gamma+1,\gamma} \mid \gamma < \zeta \rangle$. We may assume that $a_\gamma \leftrightarrow_1 b_\gamma$ for each $\gamma < \zeta$; if not, then replace each such a_γ with some a'_γ such that $a'_\gamma \leftrightarrow_0 a_\gamma$ and $a'_\gamma \leftrightarrow_1 b_\gamma$. This is possible by Proposition 4.32 and the elementarity of the structures N_ξ, since b_γ has the

desired properties. This change only involves finitely many of the functions a_γ, so the condition obtained from s by making this substitution is still in $[s]$.

Now pick $s' \leqslant^* s$ in D. Because of the assumption we made on s, Proposition 4.24 implies that there is $b' \leftrightarrow a^{s',\zeta}$ such that $([s'], b') \leqslant ([s], b)$.

The proof for clause(2) is similar. Fix $([s], b) \in R$, and assume that $a^{s,\zeta}_{\gamma+1,\gamma} \leftrightarrow_1 b_\gamma$ for all $\gamma < \zeta$. Now fix $\mu < \omega_1$ so that $\{b, \eta\} \subset N_\mu$ and extend b to $b' \in N_\mu$ by setting $b'_\nu(\xi) = \eta$ for some ξ which is not in the domain of any function in s. Then there is $a'_\nu \supset a_\nu$ so that $a'_\nu \leftrightarrow_0 b_\nu$. Now extend s to s' by setting $a^{s',\zeta}_{\gamma',\gamma} = a'(\xi)$ for all $\gamma' \in (\gamma, \zeta]$. $\hfill\square$

The ordering $(P(\vec{N})/\leftrightarrow, \leqslant^*)$ is not countably complete: it is easy to find an infinite descending sequence of conditions $\langle [s_n] \mid n < \omega \rangle$ such that any lower bound would require an ultrafilter concentrating on non-well founded sets of ordinals. However the partial order R is countably complete due to the guidance of the second coordinate b:

Lemma 4.34. *The partial order R is countably closed.*

Proof. Suppose that $\langle ([s_n], b_n) \mid n < \omega \rangle$ is a descending sequence in R. We define a lower bound $([s_\omega], b_\omega)$ for this sequence. The definition of R determines $b_{\omega,\nu} = \bigcup_{n<\omega} b_{n,\nu}$, and determines all of s_ω except for the functions $a^\omega_\nu = a^{s_\omega,\zeta}_{\nu+1,\nu}$. It also determines $\mathrm{domain}(a^\omega_\nu) = \mathrm{domain}(b_{\omega,\nu}) = \bigcup_{n<\omega} \mathrm{domain}(a^{s_n,\zeta}_{\zeta,\nu})$. Pick any $\vec{n} = \langle n_\nu \mid \nu < \zeta \rangle \in \mathcal{N}$, and for each $\nu < \zeta$ pick $a^\omega_\nu \in N_\nu$ so that

$$a^\omega_\nu \restriction \mathrm{domain}(a^n_\nu) \leftrightarrow_{k_{n,\nu}} a^n_\nu \quad \text{and} \quad a^\omega_\nu \leftrightarrow_{n_\nu} b_{\omega,\nu}$$

where $a^n_\nu \leftrightarrow_{k_{n,\nu}} b_{n,\nu}$. This is possible by the elementarity of the models N_ξ, since $b_{\omega,\nu}$ satisfies these conditions. Then $([s_\omega], b_\omega) \in R$ and $([s_\omega], b_\omega) \leqslant ([s_n], b_n)$ for each $n \in \omega$. $\hfill\square$

We are now ready to construct the desired M_B-generic set $G \subset i_\Omega(P(\vec{E}\restriction\zeta)/\leftrightarrow)$, where $\zeta = \mathrm{otp}(B)$.

Definition 4.35 (The generic set G). Let $H \subset R$ in $V[h]$ be an M-generic set. Such a set can be constructed in $V[h]$ using Lemma 4.34, since $|M|^{V[h]} = \omega_1$ and and $^\omega M \subseteq M$.

We set

$$G = \{\, [s'] \mid (\exists([s], b) \in H)(\exists \vec{\gamma} \in [\zeta]^{<\omega})\, s' \geqslant^* \mathrm{add}(i_\Omega(s), \vec{w}(s, b, \vec{\gamma})) \,\} \qquad (11)$$

where $\vec{w}(s, b, \vec{\gamma})$ is defined as follows: Set $n = \mathrm{length}(\vec{\gamma})$. Then $\vec{w}(s, b, \vec{\gamma}) = \langle i_{\gamma_i}(w_i) \mid i < n \rangle$, where

$$w_i \restriction [0, \gamma_i] = \mathrm{add}(s, w_i) \restriction [0, \gamma_i] \qquad\qquad \text{and}$$
$$a^{w_i}_{\gamma,\gamma_i} = b_{\gamma,\gamma_i} \restriction \mathrm{domain}(a^{s,\gamma_i}_{\gamma,\gamma_i}) \qquad\qquad \text{for } \zeta \geqslant \gamma > \gamma_i. \qquad (12)$$

Note that $w_i \leftrightarrow s{\upharpoonright}\gamma_i$ and therefore $[\mathrm{add}(i_\Omega(s), i_{\gamma_i}(w(s, b, \vec{\gamma})))] \leqslant [i_\Omega(s)]$. The effect of the substitution used in equation (12) to define w_i is that

$$[\mathrm{add}(i_\Omega(s), i_{\gamma_i}(w_i))] \Vdash h_{\zeta,\gamma_i}(\xi) = b_{\gamma_i}(\xi) \qquad \text{for all} \qquad \xi \in \mathrm{domain}(a_{\zeta,\gamma_i}^{s,\zeta}).$$

In looking at the Chang model inside of $M_B[G]$, it is important to recall that the set T terms specified for the sharp of $\mathbb{C}$ provides a set, inside M, of names for the members of $\mathbb{C}_B$. Definition 4.37 below makes this more specific, and provides a set of names inside M for the members of M_B and for $\mathbb{C}^{M_B}$, and then provides standard *forcing* names which are useful inside $M_B[G]$; however the notation in the next definition is sometimes useful.

Definition 4.36. We write $\bar{i}_\gamma$ for the embedding $i_{\gamma'}$ where γ' is the ordinal such that the γth member $\bar{\kappa}_\gamma$ of B is equal to $\kappa_{\gamma'}$.

If τ is an expression then we write $\ulcorner \tau \urcorner$ to indicate that τ is being used as a name for the value of the expression.

Definition 4.37. A *standard name* for a member of M_B is a term obtained recursively as follows:

1. If $\gamma \leqslant \zeta$ and $\bar{\beta} \in [\kappa, \kappa^{+\omega_1})$ then $\ulcorner \bar{i}_\gamma(\bar{\beta}) \urcorner$ is a standard name for the generator $\beta = \bar{i}_\gamma(\bar{\beta})$ belonging to $\bar{\kappa}_\gamma$.

2. If $f \in M$ and x is a finite sequence of standard names of generators β_i in M_B, then $\ulcorner i_\Omega(f)(x) \urcorner$ is a standard name for the value $i_\Omega(f)(\vec{\beta})$.

A standard name for a member of $\mathbb{C}$ is a term obtained recursively using clause (1) above and the following two operations:

2′. If α is an ordinal, then a standard name for $\alpha \in M_B$ from clause (2) above is also a standard name for $\alpha \in \mathbb{C}$.

3. Suppose that i is a standard name for an ordinal ι and that $\vec{\tau}$ is a countable sequence of standard forcing names for ordinals $\vec{\beta} = \langle \beta_k \mid k \in \omega \rangle$. Then $\ulcorner \{ x \in \mathbb{C}_i \mid C_i \models \varphi(x, \vec{\tau}) \} \urcorner$ is a standard name for $\{ x \in C_\iota \mid C_\iota \models \varphi(x, \vec{\beta}) \}$.

The definition of a *standard forcing name* is identical in both cases, except that clause 1 is replaced with the following:

1′. Suppose $([s], b) \in H$, $\xi \in \mathrm{domain}(a_{\zeta,\gamma}^{s,\gamma})$ and $b_\gamma(\xi) = \bar{\beta}$, so that

$$([s], b) \Vdash_R M_B[G] \models h_{\zeta,\gamma}(\xi) = i_\gamma(\bar{\beta}).$$

Then $\ulcorner h_{\zeta,\gamma}(\xi) \urcorner$ is a standard forcing name for $\beta = i_\gamma(\bar{\beta})$, and is said to be *established* by the condition $([s], b)$.

An arbitrary standard forcing name τ is established by $([s], b)$ if this condition establishes all names $\ulcorner h_{\zeta,\gamma}(\xi) \urcorner$ occurring in τ.

Claim 4.37.1. G is M_B-generic.

Proof. Let $D \subseteq i_\Omega(P(\vec{E}\restriction\zeta)/\leftrightarrow)$ be dense, and let

$$\dot{D} = i_\Omega(d)(\langle\, h_{\zeta,\gamma_i}(i_\Omega(\xi_i)) \mid i < k \,\rangle)$$

be a standard forcing name for D, established by a condition $(s, b) \in R$. Thus for any $\vec{w} \in \prod_{i<k} A_{\gamma_i}^{s,\zeta}$, the condition $\mathrm{add}(s, \vec{w})$ decides the values of each of the $(P(\vec{E}\restriction\zeta)/\leftrightarrow)$-names $h_{\zeta,\gamma_i}(\xi_i)$ and hence determines the value of $d(\langle\, h_{\zeta,\gamma_i}(\xi_i) \mid i < k \,\rangle) \subseteq P(\vec{E}\restriction\zeta)/\leftrightarrow$. We write $d(\vec{w})$ to denote this value.

Since D is dense,

$$A = \{\, \vec{w} \in \prod_{i<k} A_{\gamma_i}^{s,\zeta} \mid d_s(\vec{w}) \text{ is dense in } P(\vec{E}\restriction\zeta)/\leftrightarrow \,\} \in \prod_{i<k} U_{\gamma_i}^{s,\zeta}$$

so we can assume that $d_s(\vec{w})$ is dense in $P(\vec{E}\restriction\zeta)/\leftrightarrow$ for all $\vec{w} \in \prod_{i<k} A_{\gamma_i}^{s,\zeta}$. Using Lemma 4.16(2) and Lemma 4.14(2), it can be shown that there is $s' \leqslant^* s$ such that

$$(\forall \vec{w} \in \prod_{i<k} A_{\gamma_i}^{s',\zeta})(\exists e \in [\zeta]^{<\omega})(\forall t \leqslant s')\ \big(e \subseteq \mathrm{domain}(t) \implies [t] \in d_s(\vec{w})\big).$$

Since $[\zeta]^{<\omega}$ is countable, we can furthermore assume that e does not depend on $\vec{w}$. But now we are done, for if b' is such that $(s', b') \leqslant (s, b)$ in R and $e \subseteq \vec{\gamma}$ then $\mathrm{add}(i_\Omega(s'), \vec{w}(s', b', \vec{\gamma})) \in D \cap G$. $\qquad\square$

This completes the proof of Lemma 4.30.

4.6.2 Defining $\mathbb{C}_B$ in $M[G]$

It follows immediately from the genericity of G that

Corollary 4.38. $\mathbb{C}_B = \mathbb{C}^{M_B[G]}$ *for any suitable sequence B.* $\qquad\square$

Here $\mathbb{C}^{M_B[G]} = \mathbb{C}_B^{M_B[G]}$ is the set defined inside $M_B[G]$ using the definition of $\mathbb{C}$ given in the first paragraph of this paper. The more important case of a limit suitable set B is more delicate since $M_{\tilde{B}}$ is not definable inside $M_B[G]$ for suitable $\tilde{B} \subset B$. The following is the promised precise definition of $\mathbb{C}_B$:

Definition 4.39. Suppose B is a limit suitable set, and let $B' \subset B$ be the set of heads of gaps in B. Call a countable set $v \in M_B$ of ordinals B-*bounded* if for all $\lambda \in B'$ and $f \colon [\Omega]^{<\omega} \to \Omega$ in M_B, the set $f[[v]^{<\omega}] \cap \lambda$ is bounded in $M_B \cap \lambda$. Let $\mathcal{C}$ be the set of B-bounded sets. Then $\mathbb{C}_B$ is the set $L_\Omega^{M_B[G]}(\mathcal{C})$, constructed by recursion over the ordinals in $M_B \cap \Omega$ as in the first paragraph of this paper using countable sequences from $\mathcal{C}$.

Note that $\mathbb{C}_B$ is definable inside $M_B[G]$. The following Proposition implies that Definition 4.39 is equivalent to the more informal one given in section 3.3.

Proposition 4.40. *A countable sequence $\vec{\nu}$ of ordinals in M_B is B-bounded if and only if there is a suitable $\tilde{B} \subset B$ such that $\vec{\nu} \in M_{\tilde{B}}$.*

Proof. It is easy to see that if $\tilde{B}$ is suitable then every countable $\vec{\nu} \subset M_{\tilde{B}}$ is B-bounded. For the converse, suppose that $\vec{\nu}$ is B-bounded and take for each $\nu_k \in \vec{\nu}$ a function $g_k \in M$ and finite sequence of generators e_k for cardinals in B such that $\nu_k = i_\Omega(g_k)(e_k)$; taking for each k the least possible sequence e_k in the usual well order of finite sets of ordinals: $e' \prec e \iff \max((e \cup e')\setminus(e \cap e')) \in e$.

Now let f_k be the pseudoinverse of g_k defined by setting $f_k(\nu)$ equal to the $\prec$-least finite sequence e such that $\nu = g_k(e)$. Then every member of $i_\Omega(f_k)(\nu_k)$ is a generator for some member of B, for otherwise let ξ be the largest counerexample, $\xi = \max(i_\Omega(f_k)(\nu_k)\setminus B)$. Then there is a function $h \in M$ and a set $e'' \subset \xi$ of generators for members of B such that $\xi = i_\Omega(h)(e'')$, but $e'' \cup f_k(\nu_k)\setminus\{\xi\} \prec f_k(\nu_k) \leq b_k$, contradicting the minimality of b_k.

Now $\vec{\eta} = \bigcup_{k \in \omega} f_k(\nu_i)$ is B-bounded: suppose to the contrary that $f[\vec{\eta}]$ is unbounded in $\lambda \cap M_B$ where λ is the head of a gap in B. Then $f \circ g[\vec{\nu}]$ is also unbounded in λ, where $g(\nu) = \sup_{k \in \omega}(f_k(\nu) \cap \lambda)$, and this contradicts the assumption that $\vec{\nu}$ is B-bounded. Finally, the set of $\lambda \in B$ which have a generator in $\vec{\eta}$ is also B-bounded, and it follows that it is contained in a suitable subset $\tilde{B} \subset B$. □

4.7 Proof of the Main Lemma

The purpose of this subsection is to prove Lemma 3.13 under the additional assumption that $\kappa = \kappa_0$ is a member of the limit suitable set B. The following Subsection 4.8 will complete the proof of Lemma 3.13, and hence of Theorem 1.5, by removing this assumption. In the process it wil indicate the technique used to prove the stronger result Theorem 3.8.

Before beginning the proof, we state two general facts about iterated ultrapowers. Both are well known facts, but we need to verify that they are valid in the context in which they will be used.

A full statement of the conditions under which these properties hold is somewhat delicate, so we will restrict consideration to the iterated ultrapowers needed here. If k and k' are iterated ultrapowers, then we write $k[k']$ for the copy map, that is, the direct limit of the maps $i^{k(U)}$ where $U = (F)_b$ for some extender F used in the iteration k' and some generator b for F. Every extender F used satisfies $k(F) = k[F]$ for any iteration k such that $\mathrm{crit}(F)$ is not moved. In the following the term *extender* means an extender with this property which does not overlap any measurable cardinals.

Lemma 4.41. *Suppose $\kappa' \leqslant \kappa$, E' is an extender on κ', and E is an extender on κ. Suppose further that if $\kappa' = \kappa$ then $E' \lhd E$. Then the following diagram commutes:*

$$
\begin{array}{ccc}
\mathrm{Ult}(V, E) & \xrightarrow{\ i^E[i^{E'}]\ } & \mathrm{Ult}(V, E \times E') \\[2pt]
{\scriptstyle i^E}\Big\uparrow & \overset{i^{E \times E'}}{\dashrightarrow} & {\scriptstyle i^E}\Big\uparrow \\[2pt]
V & \xrightarrow[\ i^{E'}\ ]{} & \mathrm{Ult}(V, E')
\end{array}
\tag{13}
$$

Proof. The diagram (13) is the direct limit of the same diagram for the ultrafilters $(E)_a$ and $(E')_b$, where a and b are generators of E and E' respectively. $\quad\square$

Corollary 4.42. *Any iteration can be rearranged to an equivalent iteration with strictly increasing critical points.* $\hfill\square$

The second statement is a variant of Kunen's result in [7] that for any ordinal α there are at most finitely many cardinals having a measure U such that $i^U(\alpha) > \alpha$. The statement of the following lemma is tailored to its use in the proof of the main Lemma:

Lemma 4.43. *Suppose that b is a finite subset of I, $B \subset I$ is suitable, and k is an iteration in $M_\Omega[B]$ of length less than ω_2 which uses only extenders of the form $i_\nu((E)_\alpha)$ where $\kappa_\nu \in B\backslash b$ and $\alpha < \omega_1$. Then $k{\upharpoonright}(\Omega \cap M_b)$ is the identity. .*

The proof uses the following lemma. We write $\mathrm{Crit}(k)$ for the set of critical points of the extenders in the iteration k. Note that the hypothesis implies that $k'[k] = k'(k)$ for any iteration k' which is the identity on $\mathrm{Crit}(k)$.

Lemma 4.44. *Suppose $b \subseteq I$ is finite and $\alpha \in M_b$. Then there is a sequence $\vec{\nu} = \langle\, \nu_\lambda \mid \lambda \in b \cup \{0\} \,\rangle$ in M_b satisfying $(\forall \lambda \in b)\ \lambda \leqslant \nu_\lambda < \min(\{\Omega\} \cup b\backslash\lambda + 1)$ which has the following property: Let $k \in M_\Omega$ be any iteration of length less than κ_0 such that $\mathrm{Crit}(k) \cap [\lambda, \nu_\lambda] = \varnothing$ for all $\lambda \in \{0\} \cup b$. Then $k(\alpha) = \alpha$.*

Note that the statement of this lemma is first order, and hence it is also valid (using the image of the same sequence $\vec{\nu}$) in any iterated ultrapower of M_Ω.

Proof. The proof closely follows that of Kunen. We will work inside M_Ω, but the fact that $M_b \prec M_\Omega$ ensures that the ordinals ν_λ are members of M_b.

We will suppose that the lemma is false for b and α. Set $\bar{b} = \{0\} \cup b \cap \tau$, where $\tau \in b$ is least such that there is no sequence $\langle\, \nu_{\lambda'} \mid \lambda \in \{0\} \cup b \cap \tau \,\rangle$ which satisfies the conclusion for iterations k with $\mathrm{Crit}(k) \subset \tau$. Note that $\tau \leqslant \max(b \cap \alpha)$, since $\nu_{\max(b\cap\alpha)}$ can be α. Set $\bar{\tau} = \max(\bar{b})$, let $\langle\, \nu^0_\lambda \mid \lambda \in \bar{b} \cap \bar{\tau} \,\rangle$ witness that τ is minimal, and set $\nu^0_{\bar\tau} = \mathrm{cf}(\alpha)$ if $\max(\bar{b}) \leqslant \mathrm{cf}(\alpha) \leqslant \max(b)$, and $\nu^0_{\bar\tau} = \bar{\tau}$ otherwise. Following Kunen, the failure of the lemma implies that there is an infinite sequence $\langle\, \kappa_n \mid n \in \omega \,\rangle$ of iterations such that

$$(\forall n \in \omega)\ k_n(\alpha) > \alpha,$$
$$(\forall \lambda \in \bar{b})\ \min(\mathrm{Crit}(k_0))\backslash\lambda > \nu^0_\lambda, \qquad \text{and}$$
$$(\forall \lambda \in \bar{b})(\forall n \in \omega)\ \min\big(\mathrm{Crit}(k_{n+1})\backslash\lambda\big) > \sup\big(\mathrm{Crit}(k_n) \cap \min(b\backslash\lambda + 1)\big).$$

Now set $k'_{0,1} = k_0\colon M_\Omega = N_0 \to N_1$ and $k'_{n,n+1} = k'_n[k_n]\colon N_n \to N_{n+1}$. Then the direct limit N_ω of these iterations is well founded; however the following claim implies that $\langle\, k'_{n,\omega}(\alpha) \mid n \in \omega \,\rangle$ is strictly descending. This contradiction will complete the proof of Lemma 4.44.

Claim 4.44.1. $k'_{n,n+1}(\alpha) > \alpha$ *for each $n \in \omega$.*

Proof of Claim 4.44.1. Set $\ell = k'_n$ and $\ell' = k_{n+1}$, and write $\ell = \ell_1 \circ \ell_0$ and $\ell' = \ell'_1 \circ \ell'_0$, where ℓ_0 and ℓ'_0 use the extenders below $\bar{\tau}$, while ℓ_1 and ℓ'_1 use the extenders above $\bar{\tau}$. Now consider the following diagram, which is obtained using Corollary 4.42.

$$
\begin{array}{c}
\text{(diagram)}
\end{array}
\tag{14}
$$

The choice of $\langle \nu^0_\lambda \mid \lambda \in \bar{b} \rangle$ implies that $h_0(\alpha) = \alpha$, so

$$\ell_0[\ell'_1](\alpha) = \ell_0[\ell'_1] \circ \ell'_0[\ell_0](\alpha) = \ell'[\ell_0] \circ \ell'_1(\alpha) \geqslant \ell'_1(\alpha) = \ell'(\alpha) > \alpha.$$

We will embed $\ell_0[\ell'_1](\alpha)$ into $\ell'_0[\ell_1](\ell'_1)(\alpha)$, showing that the latter is also greater than α. To this end, let g and γ be a function in M^h and a generator of $\ell_0[\ell'_1]$ such that $\ell_0[\ell'_1](g)(\gamma) < \ell_0[\ell'_1](\alpha)$. We will define a function $\bar{g} \in M^{\ell_1, h_0}$, and the desired embedding will be given by $\ell_0[\ell'_1](g)(\gamma) \mapsto \ell_1[\ell'_1](\bar{g})(\ell'_0[\ell_1](\gamma))$.

For each $\nu \in \operatorname{domain}(g)$, let the function f_ν and the generator β_ν of $\ell'_0[\ell_1]$ be such that $g(\nu) = \ell'_0[\ell_1](f_\nu)(\beta_\nu)$. Note that $\ell'_0[\ell_1] \in M^{h_0}$, so the function $h(\nu, \xi) = f_\nu(\xi)$ is also in M^{h_0}. Also $\langle \beta_\nu \mid \nu \in \operatorname{domain}(g) \rangle \in M^{h_0}$, and since $\sup(\operatorname{Crit}(\ell'_0[\ell_1])) < \min(\operatorname{crit}(\ell_0[\ell'_1]))$, there is some β such that $\beta_\nu = \beta$ for almost all ν; that is, $\gamma \in \ell_0[\ell'_1](\{ \nu \mid \beta_\nu = \beta \})$.

Now set $\bar{g}(\xi) = \ell'_0[\ell_1](h)(\beta, \xi)$, so $\bar{g}(\ell'_0[\ell_1](\nu)) = g(\nu)$ for almost all ν.

This completes the proof of Claim 4.44.1 and hence of Lemma 4.44. $\square$

Proof of Lemma 4.43. We will show that for any finite $b \subseteq I$ and $\alpha \in M_b$ the sequence $\vec{\nu}$ given by Lemma 4.44 is also valid for iterations k as in Lemma 4.43. Note that such k, having all critical points in $M_{B \setminus b}$, satisfy the constraint given by $\vec{\nu}$.

Supposing the contrary, let b be a sequence for which the claim fails, let α the least ordinal for which it fails, and let $k \in M_B$ witness this failure. Set $\zeta = \operatorname{otp}(B)$, and let $G \subset i_\Omega(P(\vec{E} \restriction \zeta)/\leftrightarrow)$ be the generic set constructed in Subsection 4.6, so that $k \in M_\Omega[G]$. Then there is a condition $s \in G$ such that $\{ \bar{\kappa}^{s, \nu} \mid \nu \in \operatorname{domain}(s) \} \subseteq b \cup \{\Omega\}$ which forces that α is the least counterexample and that $\dot{k}$ is a name for a witness to this failure.

The choice of $\vec{\nu}^0$ ensures that k is continuous at α, and therefore there is some $\alpha' < \alpha$ such that $k(\alpha') \geqslant \alpha$. By Lemma 4.16(2) there is a condition $s'' \leqslant^* s$ in G and a finite $e \subseteq \zeta$ such that any $s' \leqslant s$ with $e \subseteq \operatorname{domain}(s'')$ determines α'. Fix $s' \leqslant s''$ in M_b with $e \subseteq \operatorname{domain}(s')$ and $\nu_\lambda < \bar{\kappa}^{s', \nu}$ whenever $\lambda \in b \cup \{0\}$ and $\lambda < \bar{\kappa}^{s', \nu}$.

Now let $j\colon M_\Omega \to M^j$ be the iteration of M_Ω by the extenders

$$\langle\, F_\xi^{s',\nu} \mid \nu \in \mathrm{domain}(s')\backslash\{\Omega\} \wedge \bar\kappa^{s',\nu} \notin \lim(B) \wedge \xi \in \mathrm{domain}(\vec{F}^{s',\nu})\,\rangle. \qquad (15)$$

Now construct, as in Subsection 4.6 (except that the second component $\vec{b}$ of the conditions of R is modified appropriately), $G' \subseteq j \circ i_\Omega(P(\vec{E}\!\restriction\mathrm{otp}(B))/\hookleftarrow)$ with $s' \in G'$. Instead of taking all indiscernibles from I, this construction uses the iteration $j \circ i_\Omega$, substituting the critical point of $F_\xi^{s,\nu}$ for the corresponding member of B whenever $F_\xi^{s,\nu}$ is in the sequence (15).

Now factor $\dot{k}^{G'}$ as $\ell_1 \circ \ell_0$ where ℓ_0 uses the extenders of $\dot{k}^{G'}$ which are in M_b and ℓ_1 uses the remainder. Note that since M_b is closed under countable sequences, $\ell_0 \circ j \in M_b$, and since $\ell_0 \circ j$ obeys $\vec{\nu}$ it follows that $\ell_0 \circ j(\alpha) = \alpha$. Therefore $\ell_0 \circ j(\alpha') < \alpha$, but $(\ell_1 \circ \ell_0)(j(\alpha')) \geqslant j(\alpha) = \alpha$, so $\ell_1(\ell_0 \circ j(\alpha')) > \ell_0 \circ j(\alpha')$. Since the map j is elementary, this contradicts the minimality of α. $\qquad\square$

This concludes the preliminary observations, and we are now ready to continue with the proof of the Main Lemma, 3.13. As was stated earlier, this proof is an induction on the lexicographic ordering of pairs (ι, φ) in order to prove that for all limit suitable sequences B and all x in $\mathbb{C}_\iota \cap M_B$,

$$\mathbb{C}_B \mathrel{\vDash_{\overline{\mathbb{C}_\iota}}} \varphi(x) \quad \text{if and only if} \quad \mathrel{\vDash_{\overline{\mathbb{C}_\iota}}} \varphi(x). \qquad (16)$$

Here and for the remainder of the paper we write $P \mathrel{\vDash_{\overline{\mathbb{C}_\iota}}} \sigma$ to mean that $(\mathbb{C}_\iota)^P \models \sigma$.

The statement (16) uses the induction hypothesis: $\mathbb{C}_B$ is not, by its definition, a subset of $\mathbb{C}$; however by the induction hypothesis there is an embedding $\pi\colon (\mathbb{C}_\iota)^{\mathbb{C}_B} \to \mathbb{C}_\iota$, which is the identity on ordinals and is defined in general by setting $\pi(\{\, y \in (\mathbb{C}_{\iota'})^{M_B} \mid (\mathbb{C}_{\iota'})^{M_B} \models \varphi(y,a)\,\}) = \{\, y \in \mathbb{C}_{\iota'} \mid \mathbb{C}_{\iota'} \models \varphi(y,\pi(a))\,\})$. For the rest of this section we will identify $(\mathbb{C}_\iota)^{M_B}$ with the range of π.

We will need an additional induction hypothesis in order to carry out the proof of Lemma 3.13. This hypothesis is rather technical and uses notation which will be developed during the proof of the induction step for Lemma 3.13, so we defer its statement, as Lemma 4.50, until it is needed to complete that proof.

By standard arguments, the only problematic part of the proof of the induction step for Lemma 3.13 is the assertion that the existential quantifier is preserved downwards: We assume that $\psi(x,y)$ is a formula which satisfies (16), and want to prove that

$$\forall x \in \mathbb{C}_B \left(\mathrel{\vDash_{\overline{\mathbb{C}_\iota}}} \exists y\,\psi(x,y) \implies \mathbb{C}_B \mathrel{\vDash_{\overline{\mathbb{C}_\iota}}} \exists y\,\psi(x,y) \right). \qquad (17)$$

Since the basic problem in the proof is dealing with gaps in B, it will be helpful to introduce some terminology to describe their structure. A *gap* of B is a maximal nonempty interval of $I\backslash B$. For a limit suitable set B, the gap

will be a half open interval $[\sigma, \delta)$ where σ is the supremum of an ω-sequence of members of B, and δ is either $\min(B \backslash \sigma)$ or Ω. We call δ the *head* of the gap.

Let $\delta' = \sup(\sigma \cap I \backslash B)$, or $\delta' = 0$ if $I \cap \sigma \subseteq B$. Then $[\delta', \sigma) \cap I \subseteq B$; we refer to this interval as the *block* of B corresponding to the gap, and to δ' (which either is 0 or is also the head of a gap below δ') as the *foot* of the block. If $\sigma' = \sup((B \cap \lim(I)) \cap \delta)$ then $B \cap (\sigma', \sigma) = I \cap (\sigma', \delta)$ is an ω sequence of successor members of B; we will refer to this interval as *the tail* of the gap. If γ is any member of this tail then we will refer to the interval $[\gamma, \sigma) \cap B$ as *the tail of B above γ*.

Call a set $b \subseteq B$ a *tail traversal* of B if it contains exactly one point from the tail of each gap in B. Then b determines a suitable subsequence $\tilde{B} \subseteq B$ as follows: let δ be the head of a block in B, let δ' be the foot of the associated block, and let γ be the unique member of $b \cap [\delta', \delta)$. Then we regard γ as dividing this block of B into three parts: the closed interval $[\delta', \gamma) \cap B$, which we will call a *closed block of B below γ*, the singleton $\{\gamma\}$, and the tail $(\gamma, \delta) \cap B$, which we will call the *tail above b*. The suitable subsequence $\tilde{B}$ determined by b is the union of the closed blocks of B below the members of b.

The maximal suitable subsequences of B are those which are determined by some tail traverse of B. Note that any suitable subsequence of B is contained in a maximal subsequence, and hence in dealing with $\mathbb{C}_B$ we only need to consider maximal suitable subsequences.

We are now ready to begin the proof of the induction step for Lemma 3.13. Suppose that $\varphi(x)$ is the formula $\exists y \, \psi(x, y)$ and is true in $\mathbb{C}_\iota$, and that B is a limit suitable sequence with $x \in \mathbb{C}_B$. Fix a tail traversal b of B such that $\{x, \iota\} \subseteq \mathbb{C}_{\tilde{B}}$, where $\tilde{B}$ is the suitable subsequence of B determined by b. Pick y so $\models_{\overline{\mathbb{C}_\iota}} \psi(x, y)$ and let $B' \supseteq B$ be a limit suitable sequence with $y \in \mathbb{C}_{B'}$. By the induction hypothesis $\mathbb{C}_{B'} \models_{\overline{\mathbb{C}_\iota}} \psi(x, y)$.

We will define an iteration map k and an isomorphism σ as in Diagram (18).

$$
\begin{array}{ccc}
M_{B'} & \longleftarrow & M_{B'} \upharpoonright \vec{\eta} \\
\big\uparrow & & \big\updownarrow \sigma \\
M \xrightarrow{i_\Omega} M_{\tilde{B}} \hookrightarrow M_B \xrightarrow{\;k\;} M_k & \longleftarrow & M_k \upharpoonright \vec{\eta}
\end{array}
\tag{18}
$$

The map k will be an iterated ultrapower using iterated extenders with critical points in b. It has length greater than ω_1, but is definable in $M_B[c]$ from a countable sequence $c \in M_B$ of ordinals. The iteration k has two purposes:

1. It includes one iteration step for each member of $B' \backslash \tilde{B}$ (excluding a tail in B' of each gap of B).

2. For each gap in B' which does not correspond to a gap of B, it includes an ω_1-sequence of iteration steps inserted in order to emulate this gap inside M_k.

The submodel $M_k \restriction \vec\eta$ of M_k will be obtained by using only the iterations from clause 1, omitting those from clause 2. The isomorphism σ will map members of $B' \setminus B$ to the corresponding critical points of ultrapowers in clause 1, and the submodel $M_{B'} \restriction \vec\eta$ of $M_{B'}$ will be obtained by taking only the generators belonging to members of $B' \setminus B$ which correspond to generators of extenders used in the iteration steps from clause 1.

The iteration k will be such that Lemma 4.43 implies that the restrictions of k and σ to ordinals in the suitable submodel $M_{\tilde B}$ are the identity. The iteration k can be defined in $M_B[c]$, for a countable sequence c of ordinals, and thus is definable in the extension $M_B[G]$. The models M_B and M_k have the same ordinals and the same associated Chang model $\mathbb{C}_B = \mathbb{C}_k$. Thus Diagram (18) induces the following diagram:

$$
\begin{array}{ccc}
\mathbb{C}_{B'} & \longleftarrow & \mathbb{C}_{B'}\restriction\vec\eta \\
\big\uparrow & & \Big\updownarrow \sigma \\
\mathbb{C}_{\tilde B} \hookrightarrow \mathbb{C}_B & \xrightarrow{\ k\ } \mathbb{C}_k = \mathbb{C}_B & \longleftarrow \mathbb{C}_k\restriction\vec\eta
\end{array}
\qquad (19)
$$

Once this machinery has been put into place, we will be able to complete the proof of the induction step for Lemma 3.13: we are assuming $\models_{\mathbb{C}_\iota} \psi(x,y)$, with x and y in $\mathbb{C}_{B'}$, so by the induction hypothesis $\mathbb{C}_{B'} \models_{\mathbb{C}_\iota} \exists y \psi(x,y)$. An easy proof will give Lemma 4.47, which implies that $\mathbb{C}_{B'}\restriction\vec\eta \prec \mathbb{C}_{B'}$, so $\mathbb{C}_{B'}\restriction\vec\eta \models_{\mathbb{C}_\iota} \exists y \psi(x,y)$. Fix $y \in \mathbb{C}_{B'}\restriction\vec\eta$ so that $\mathbb{C}_{B'}\restriction\vec\eta \models_{\mathbb{C}_\iota} \psi(x,y)$. Since σ is an isomorphism, $\mathbb{C}_k\restriction\vec\eta \models_{\mathbb{C}_\iota} \psi(x,\sigma(y))$.

Now we want to conclude that $\mathbb{C}_B \models \psi(x,\sigma(y))$, but unlike the case in the previous paragraph, we don't know of a direct proof that $\mathbb{C}_B\restriction\vec\eta \prec \mathbb{C}_B$. Instead we will state a slightly generalized form of the needed fact as Lemma 4.50, and with this as an additional induction hypothesis conclude the proof of the induction step for Lemma 3.13. We then use the induction hypothesis (including the just proved fact that Lemma 3.13 holds for the pair (ι, φ)) to prove that Lemma 4.50 holds for (ι, φ); this will complete the proof of Lemmas 3.13 and 4.50, and thus of Theorem 1.5, except for the assumption that $\kappa_0 \in B$.

We now give the details of the construction of Diagram (18). We already have the four models on the left of the diagram: B is the given limit suitable sequence, $\tilde B \subset B$ is a suitable subsequence with $x \in M_{\tilde B}$ which is characterized by a tail traversal b of B, and $B' \supseteq B$ is a limit suitable sequence with a witness y to $\exists y\, \psi(x,y)$. The following definition is more general than needed here. The added generality is used in the proof of Lemma 4.50.

Definition 4.45. A *virtual gap construction sequence* for B is a triple $(b, \vec\eta, g)$ satisfying the following four conditions:

1. The set b is a tail traversal sequence of B.

2. $\vec\eta$ is a function with $\mathrm{domain}(\vec\eta) = \{\, (\lambda, \xi) \mid \lambda \in b \wedge \xi < \nu_\lambda \,\}$, where ν_λ is a countable ordinal for each $\lambda \in b$.

3. $g \subset \operatorname{domain}(\vec{\eta})$, and if $(\lambda, \xi) \in g$ then ξ is a limit ordinal.

4. Define an order $\prec$ on $B \cup \operatorname{domain}(\vec{\eta})$ using the ordinal order on B, the lexicographic order on $\operatorname{domain}(\vec{\eta})$, and setting $\lambda' \prec (\lambda, \xi) \prec \lambda$ when $\lambda' < \lambda \in B$ and $(\lambda, \xi) \in \operatorname{domain}(\vec{\eta})$.

   Then $\eta_{\lambda, \xi} > \operatorname{otp}(\{\, z \in B \cup \operatorname{domain}(\eta) \mid z \prec (\lambda, \xi)\,\})$.

We will say that $(b, \vec{\eta}, g)$ is a virtual gap construction sequence for B' over B if in addition the following four conditions hold: (i) B' and B are limit suitable sequences with $B' \supset B$. (ii) B' has the same order type as $(B \cup \operatorname{domain}(\eta), \prec)$. In the following, we write $\tau \colon (B \cup \operatorname{domain}(\eta)) \to B'$ for the order isomorphism. (iii) τ is the identity on the suitable subsequence $\tilde{B}$ of B determined by b. (iv) if $\gamma \in B' \backslash B$ then $\tau(\gamma) \in g$ if and only if γ is the head of a gap in B'.

Note that if $(b, \vec{\eta}, g)$ is a virtual gap construction sequence for B' over B then $b' = \tau^{-1}[b]$ is a traversal of the tails in B' of the gaps of B, and that if $\lambda \in b'$ then τ maps the tail above λ in B' to the tail above $\tau(\lambda)$ in B.

For the construction of Diagram (18), we use the following virtual gap construction sequence $(b, \vec{\eta}, g)$ for B' over B: The function $\vec{\eta}$ is a constant function, with the constant value η to be specified later. Fix a traversal b' of the tails in B' belonging to gaps of B. Then (i) $\operatorname{domain}(\vec{\eta}) = \{\, (\lambda, \xi) \mid \lambda \in b \wedge \xi \leqslant \operatorname{otp}(B' \cap [\lambda, \lambda')) \,\}$, where λ' is the member of b' in the tail in B' of the same gap as λ, and (ii) $g = \{\, \tau(\gamma) \mid \gamma \in B' \backslash B \wedge \gamma \text{ is the head of a gap in } B' \,\}$

Definition 4.46. If $(b, \vec{\eta}, g)$ is a virtual gap construction sequence for B' over B, then $M_{B'} {\restriction} \vec{\eta} = \{\, j_\Omega(f)(a) \mid f \in M \wedge a \in [\mathcal{G}]^{<\omega} \,\}$ where $\mathcal{G}$ is the following set of generators: Let κ_ν be a member of B' and let $\beta = i_\nu(\bar{\beta})$ be a generator belonging to κ_ν. Then

$$\beta \in \mathcal{G} \iff \big(\tau(\kappa_\nu) \in B \vee (\tau(\kappa, \nu) = (\lambda, \xi) \in \operatorname{domain}(\vec{\eta}) \wedge \bar{\beta} \in \operatorname{supp}(E_{\eta_{\lambda, \xi}}))\big).$$

Note that $M_{B'} {\restriction} \vec{\eta} \prec M_{B'}$, that $M_{\tilde{B}} \subseteq M_{B'} {\restriction} \vec{\eta}$ and, that if η is chosen sufficiently large then $y \in M_{B'} {\restriction} \vec{\eta}$. This is the first of two criteria for the choice of η; the other is that $\eta > \omega^\omega \cdot \operatorname{otp}(B')$.

Lemma 4.47. *If $(b, \eta_{\lambda, \xi}, g)$ is a virtual gap construction sequence for B' over B then $\mathbb{C}_{B'} {\restriction} \vec{\eta} \prec \mathbb{C}_{B'}$.*

Proof. The construction of Subsection 4.6 can be carried out to obtain a $M_{B'} {\restriction} \vec{\eta}$-generic subset $G \in i_\Omega(P(\vec{E} {\restriction} \operatorname{otp}(B')/{\leftrightarrow}))$. The only change needed is that the range of the coordinate b_γ in a condition of R is restricted to $\operatorname{supp}(E_{\eta_{\lambda, \xi}})$ whenever $(\lambda, \xi) \in \operatorname{domain}(\vec{\eta})$ and κ_γ is the ξth member of B' above λ.

Now let φ be a formula which is true in $\mathbb{C}_{B'} {\restriction} \vec{\eta}$. Then there is a condition $([r], b)$ in the forcing R for $M_{B'} {\restriction} \vec{\eta}$ which establishes the parameters of φ and forces φ to be true. This condition is also a condition in the forcing R for $M_{B'}$, it establishes the parameters in the same way, and it forces that φ holds in $\mathbb{C}_{B'}$. $\qquad\square$
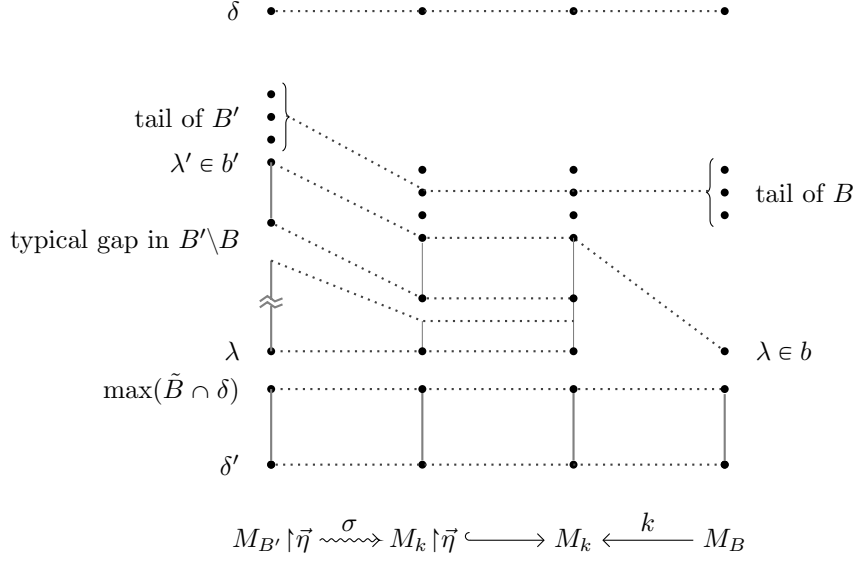
Figure 3: The maps σ and k inside the block between δ' and δ which is associated with the gap in B headed by δ. The dotted lines represent the maps σ and k; the heavier vertical lines represent intervals of I contained in the indicated models and the lighter ones represent the indiscernibles added by the iteration k.

Note that condition 4 of Definition 4.45 is used here to ensure that the enough of the image of E is present at each of the $\kappa_\nu \in B'\backslash B$ to construct the generic set as in section 4.6.

We can now complete the construction of the elements of Diagram (18) by defining k and σ. This construction is illustrated in Figure 3.

Definition 4.48. We define by recursion on $z \in (B \cup \text{domain}(\vec{\eta}), \lessdot)$ a sequence of embeddings $k_z \colon M_B \to M_z^*$. We will describe the construction on one of the blocks of B. Thus, suppose that $\delta \in B$ is the head of a gap and $\delta' \in B \cup \{0\}$ is the foot of the block of B below it. We assume that $k_z \colon M_B \to M_z^*$ has been defined for all $z \lessdot \delta'$. Let λ be the unique member of $b \cap [\delta', \delta)$.

  (i) $M_0^* = M_B$, and if $\delta' > 0$ then $M_{\delta'}^* = \text{dir}\lim\langle\, (M_z^*; k_z) : z \lessdot \delta'\,\rangle$.

  (ii) If $\nu \in \tilde{B} \cap [\delta', \delta) = B \cap [\delta', \lambda)$ then $M_\nu^* = M_{\delta'}^*$.

  (iii) If $\nu \in B \cap (\lambda, \delta)$ then M_ν^* is the direct limit of the embeddings k_z for $z \lessdot \lambda$.

  (iv) If $(\lambda, \xi) \in \text{domain}(\vec{\eta})\backslash g$ and ξ is a limit ordinal then M_ν^* is the direct limit of the embeddings k_z for $z \lessdot (\lambda, \xi)$.

  (v) If $z = (\lambda, \xi + 1) \in \text{domain}(\vec{\eta})$, or if $z = \lambda$ and (λ, ξ) is its predecessor in $\lessdot$, then $M_z^* = \text{Ult}(M_{(\lambda,\xi)}^*, E_{\eta_{\lambda,\xi}}^*)$ where, letting γ be such that $\delta' = \kappa_\gamma$, we write E_α^* for $k_{\lambda,\xi} \circ i_\gamma(E_\alpha)$.

  (vi) If $z = (\lambda, \xi) \in g$, then set $\bar{k}_z^* \colon M_B \to M_z^{**} = \text{dir}\lim_{z' \lessdot z} M_{z'}^*$. Then M_z^*

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is an iterated ultrapower of M_z^{**} of length ω_1, using extenders $\bar{k}_z^*(i_\gamma(\vec{F}))$ where $\lambda = \kappa_\gamma$ and $\vec{F} \in M$ is an arbitrary but fixed cofinal subsequence of the sequence of extenders below E on κ in M.

If $\gamma \in B'$ and $\tau(\gamma) = (\lambda, \xi) \in \mathrm{domain}(\vec{\eta})$, then $\sigma(\gamma)$ is equal to the critical point of the ultrapower of $M_{\tau(\gamma)}^*$.

Definition 4.49. The restriction of σ to B' is determined by the map τ specified in the Definition 4.45 of a virtual gap construction sequence for B' over B: if $\tau(\gamma) \in B$ then $\sigma(\gamma) = k(\tau(\gamma))$, and if $\tau(\gamma) = (\lambda, \xi)$ then $\sigma(\gamma)$ is the ξth critical point of the iteration steps of k using extenders on λ. The restriction of σ to B' determines its restriction to generators of $M_{B'} \restriction \vec{\eta}$, and this restriction determines the remainder of σ.

The particular choice of the sequence $\vec{F}$ of extenders will not matter; a suitable choice for F_ν would be the least $\kappa^{+(\nu+1)}$-strong extender on κ. It is important that $\vec{F} \in M$, for that implies that M_k is in $M_B[B, \vec{\eta}]$ and hence is in the generic extension $M_B[G]$ of M_B described in section 4.6; we use this fact to identify the ordinals of M_k with those of M_B. It is also important that $\vec{F}$ is cofinal among the extenders below E in M, and hence $i_\gamma(\vec{F})$ is cofinal among the extenders on λ in M_B: this fact ensures (using Lemma 4.43) that the restriction of k to the ordinals of M_B is independent of the choice of $\vec{F}$.

This completes the definition of the elements of Diagram (18), and the extension to the Chang model in Diagram (19) is straightforward. We have already observed that the Chang model $\mathbb{C}_k$ built on M_k is the same as $\mathbb{C}_B$, giving the identity on the bottom. Lemma 4.47 asserts that $C_{B'} \restriction \vec{\eta}$ is an elementary substructure of $\mathbb{C}_{B'}$, and $\sigma \colon C_{B'} \restriction \vec{\eta} \to \mathbb{C}_k \restriction \vec{\eta}$ is an isomorphism. It follows that $\mathbb{C}_k \restriction \vec{\eta} \models_{\mathbb{C}_\iota} \psi(x, \sigma(y))$, and we will be finished if we can conclude from this that that $\mathbb{C}_B \models_{\mathbb{C}_\iota} \psi(x, \sigma(y))$. This is implied by the case (ι, ψ) of Lemma 4.50, which is the promised addition to the induction hypothesis to be used in the proof of Lemma 3.13. Thus this concludes the proof of the induction step for Lemma 3.13.

Lemma 4.50. *Suppose that $B \subseteq B'$ are limit suitable sequences and $\vec{\eta}$ is a virtual gap construction sequence for B' over B such that $\eta_{\lambda, \xi} \geqslant \omega^n \cdot \mathrm{otp}(B \cup \mathrm{domain}(\vec{\eta}), <)$ for all $(\lambda, \xi) \in \mathrm{domain}(\vec{\eta})$ and $n < \omega$. Let $k \colon M_B \to M_k$ be the virtual gap construction iteration, and let $\mathbb{C}_k \restriction \vec{\eta} \subseteq \mathbb{C}_k$ be as given in Diagram (19). Then $\mathbb{C}_k \restriction \vec{\eta} \prec \mathbb{C}$.*

Proof. As was stated earlier, this proof is a simultaneous induction along with Lemma 3.13. We have completed the proof that Lemma 3.13 holds for (ι, φ), using as an induction hypothesis that Lemmas 3.13 and 4.50 hold for all smaller pairs. We now use this same induction hypothesis, together with the fact that Lemma 3.13 holds for (ι, φ), to prove that Lemma 4.50 holds for (ι, φ): that is, if B, k and η are as in Lemma 4.50 and x is an arbitrary member of $C_k \restriction \vec{\eta}$ such that $\models_{\mathbb{C}_\iota} \exists y \psi(x, y)$, then $\mathbb{C}_k \restriction \vec{\eta} \models_{\mathbb{C}_\iota} \exists y \psi(x, y)$. By the newly proved case of Lemma 3.13, $\mathbb{C}_B \models_{\mathbb{C}_\iota} \exists y\, \psi(x, y)$. Fix $y_0 \in \mathbb{C}_B$ so that $\models_{\mathbb{C}_\iota} \psi(x, y_0)$. We now define

an extension $\vec{\eta'}$ of the virtual gap construction sequence $\vec{\eta}$ such that $y_0 \in C_B \restriction \vec{\eta'}$. The sequence $\vec{\eta'}$ will have the same sets b and g as $\vec{\eta}$, but the domain of $\vec{\eta'}$ will be enlarged by adding an ω sequence of new elements below each $(\lambda, \xi) \in g$. Thus, for each $\lambda \in b$ define a map t_λ with $\mathrm{domain}(t_\lambda) = \mathrm{length}(\vec{\eta}_\lambda)$ by

$$
t_\lambda(\xi) = \begin{cases}
0 & \text{if } \xi = 0, \\
t_\lambda(\xi') + 1 & \text{if } \xi = \xi' + 1, \\
\sup_{\xi' < \xi} t_\lambda(\xi') & \text{if } \xi \text{ is a limit and } (\lambda, \xi) \notin g \\
\sup_{\xi' < \xi} t_\lambda(\xi') + \omega & \text{if } (\lambda, \xi) \in g.
\end{cases}
$$

Now we define $\vec{\eta'}$, using an ordinal $\eta' \in \omega_1$ to be determined shortly:

$$
\mathrm{domain}(\vec{\eta'}) = \{ (\lambda, \xi) \mid \xi < \sup \mathrm{range}(t_\lambda) \}
$$
$$
b^{\vec{\eta'}} = b^{\vec{\eta}}, \text{ and } g^{\vec{\eta'}} = \{ (\lambda, t_\lambda(\xi)) \mid (\lambda, \xi) \in g^{\vec{\eta}} \}, \text{ and}
$$
$$
\eta'_{\lambda, \xi} = \begin{cases}
\eta_{\lambda, \xi'} & \text{if } \xi = t(\xi') \\
\eta' & \text{if } (\lambda, \xi) \notin \mathrm{range}(t).
\end{cases}
$$

The first condition on η' is that $\eta' \geqslant \omega^n \cdot \mathrm{otp}(B \cup \mathrm{domain}(\vec{\eta'}, <))$ for each $n \in \omega$, and the second condition is that $y_0 \in \mathbb{C}_B \restriction \vec{\eta'}$. It is possible to satisfy the second condition since $\mathbb{C}_B = \bigcup_{\eta' < \omega_1} \mathbb{C}_B \restriction \vec{\eta'}$. Notice that the first condition implies that $\vec{\eta'}$ satisfies the hypothesis of Lemma 4.50, since if $\xi = t_\lambda(\xi')$ then $\eta'_{\lambda, \xi} = \eta_{\lambda, \xi'} > \omega^{n+1} \cdot \mathrm{otp}(B \cup \mathrm{domain}(\vec{\eta}), <) = \omega^n \cdot \omega \cdot \mathrm{otp}(B \cup \mathrm{domain}(\vec{\eta}), <) \geqslant \omega^n \cdot \mathrm{otp}(B \cup \mathrm{domain}(\vec{\eta'}), <)$.

$$(20)$$

For the remainder of the proof we refer to Diagram (20). The inner rectangle is the same as Diagram (18). The map τ is determined by using the map $(\lambda, \xi) \mapsto (\lambda, t_\lambda(\xi))$ to map the generators of indiscernibles from $\vec{\eta}$ into those of $\vec{\eta'}$. As with Diagrams (18) and (19), Diagram (20) induces a similar diagram for the corresponding Chang models.

We claim that $\tau \restriction (\mathbb{C}_k \restriction \vec{\eta})$ is the identity. First, Lemma 4.43 implies that the restriction of τ to the ordinals of $M_k \restriction \vec{\eta}$ is the identity. Now every member of

$\mathbb{C}_k \restriction \vec{\eta}$ is represented by a term $w = \{\, z \in \mathbb{C}_{\iota'} \mid \ \models_{\mathbb{C}_{\iota'}} \varphi(z, a)\,\}$, where $\iota' \in M_k \restriction \vec{\eta}$ and a is a sequence of ordinals from $M_k \restriction \vec{\eta}$. Thus $\tau(w)$ is represented by the same term in $\mathbb{C}_{k'} \restriction \vec{\eta}$. But $\mathbb{C}_k = \mathbb{C}_{k'} = \mathbb{C}_B$, so this term represents the same set w in $\mathbb{C}_{k'}$.

Now define B'' to be B' together with the next ω-many members of I from each of the gaps of B' which are not gaps of B. The right-hand trapezoid commutes, and in particular $\sigma^{-1}(x) = (\sigma')^{-1} \circ \tau(x) = (\sigma')^{-1}(x)$. Now $\mathbb{C}_{k'} \restriction \vec{\eta'} \models_{\mathbb{C}_\iota} \psi(x, y_0)$, and since σ' is an isomorphism it follows that $\mathbb{C}_{B''} \restriction \vec{\eta'} \models_{\mathbb{C}_\iota} \psi(\sigma^{-1}(x), (\sigma')^{-1}(y_0))$. It follows by Lemma 4.47 that $\mathbb{C}_{B''}$ satisfies the same formula, by the induction hypothesis Lemma 3.13 for (ι, φ) it follows that $\models_{\mathbb{C}_\iota} \exists y\, \psi(\sigma^{-1}(x), y)$, and by another application of the same induction hypothesis $\mathbb{C}_B$ satisfies the same formula. By Lemma 4.47, $\mathbb{C}_{B'} \restriction \vec{\eta}$ does as well, so let y_1 be such that $M_{B'} \restriction \vec{\eta} \models_{\mathbb{C}_\iota} \psi(\sigma^{-1}(x), y_1)$. Then $\mathbb{C}_k \restriction \vec{\eta} \models_{\mathbb{C}_\iota} \psi(x, \sigma(y_1))$, so $\mathbb{C}_k \restriction \vec{\eta} \models_{\mathbb{C}_\iota} \exists y \psi(x, y)$, as required. $\qquad\square$

4.8 Finite exceptions and $\kappa_0 \notin B$

In the last subsection we assumed that $\kappa_0 = \kappa$ is a member of B; here we indicate how this extra assumption can be eliminated. The same argument is used in the proof of Theorem 3.8 to support the provision allowing finitely many exceptions.

The reason that the previous argument fails when $\kappa_0 \notin B$ is that κ_0 may be a member of the extended model B' of diagram 18. In this case the definition of the map k in Diagram (18) fails because there is no tail of B in this first gap.

To conclude the proof of Theorem 1.5(2), suppose that $B = \{\, \lambda_\nu \mid \nu \leqslant \zeta \,\}$ is a limit suitable set with $\lambda_0 > \kappa_0$, that $x \in \mathbb{C}_B$, and that $\mathbb{C} \models \varphi(x)$. We want to show that $\mathbb{C}_B \models \varphi(x)$. Let $B' = B \cup \{\, \kappa_n \mid n < \omega \,\}$, a limit suitable sequence of length $\omega + \delta$. Since $\kappa_0 \in B'$, the version of Theorem 1.5(2) already proved implies that $\mathbb{C}_{B'} \models \varphi(x)$. Let G be the $M_{B'}$-generic subset of $i_\Omega(P(\vec{E} \restriction (\omega + \delta))/\leftrightarrow)$ constructed in section 4.6, and set

$$G_1 = \{\, [p \restriction (\omega, \omega + \delta)] \mid [p] \in G \wedge \omega \in \operatorname{domain}(p) \,\}.$$

Then G_1 is an M_B-generic subset of $i_\Omega(P'/\leftrightarrow)$, where P' is the forcing described following Lemma 4.13 such that $P(\vec{E} \restriction (\omega + \delta)) \equiv P(\vec{E} \restriction \omega) * \dot{R}$ is a regular suborder of $P(\vec{E} \restriction \omega + 1) \times P'$.

Now let $[q] \in G$ be a condition such that $[q] \Vdash \mathbb{C}_{B'} \models \varphi(x)$. We may assume that $\omega = \min(\operatorname{domain}(q))$. Let G_0 be a M_B-generic subset of $P(\vec{F}^{q, \omega})$ with $q \restriction \omega + 1 \in G_0$, and let $\tilde{G}$ be the resulting M_B-generic subset of $i_\Omega(P(\vec{E} \restriction (\omega + \delta))/\leftrightarrow)$. Then $[q] \in \tilde{G}$, so $M[\tilde{G}] \models \mathbb{C}_{B''} \models \varphi(x)$, where B'' is the set $\{\, \bar{\kappa}_n \mid n \in \omega \,\} \cup B$, interpreted as having, like B', a gap headed by λ_0. Now the forcing does add a new countable sequence of ordinals, as $M[\tilde{G}] \models \operatorname{cf}(\lambda_0) = \omega$. However, λ_0 is being interpreted as the head of a gap and therefore $\mathbb{C}_{B''} = \bigcup \{\, \mathbb{C}_{\tilde{B}} \mid \tilde{B} \subset B \wedge \tilde{B} \text{ is suitable} \,\}$. Since the forcing $P(\vec{F}^{q, \omega})/\leftrightarrow$ does not add bounded subsets of λ_0, this implies that $\mathbb{C}_{B''}$, as defined inside $M[\tilde{G}]$, is equal to $\mathbb{C}_B$. This concludes the proof that $\mathbb{C}_B \models \varphi(x)$.

It is critical to this argument that there are only a finite number of intervals (in this case, only one interval) of B which need special attention. Finitely many such special cases can be dealt with a condition q obtained, as in the proof, by finitely many one-step extensions, but infinitely many would involve adding Prikry type sequences, which requires the use of the iteration to obtain genericity.

5   Questions and Problems

This study leaves a number of questions open. Two which were mentioned in the introduction essentially involve filling gaps in this paper:

Question 5.1. Exactly what is the large cardinal strength of a sharp for $\mathbb{C}$?

Theorem 1.5 puts it between a mouse over the reals satisfying $o(\kappa) = \kappa^{+(\omega+1)} + 1$ and a sufficiently strong mouse over the reals satisfying $o(\kappa) = \kappa^{+\omega_1} + 1$. The second question asks whether this procedure truly gives a sharp for the Chang model:

Question 5.2. Can the restricted formulas be removed from the definition 1.3 of the sharp for the Chang model? That is, can the added Skolem functions be made full-fledged members of the language?

The next questions ask for more detailed information about the structure of the sharp:

Question 5.3. What is $K(\mathcal{R})^{\mathbb{C}}$? Is it an iterate (not moving members of I) of $M_{\Omega}|\Omega$ for some mouse M over the reals? If so, is this iteration definable in $L[M, \{\, \lambda \mid \mathrm{cf}(\lambda) = \omega \,\}]$?

Question 5.4. What is the core model $K^{\mathbb{C}}$ of the Chang model? How does it relate to the $K^{L(\mathcal{R})}$ and to $K(\mathcal{R})^{\mathbb{C}}$?

Question 5.5. Is it true that the measurable cardinals of $K(\mathcal{R})^{\mathbb{C}}$ are exactly the regular cardinals of $K(\mathcal{R})^{\mathbb{C}}$ which have countable cofinality in V?

The final question is about the next step from the Chang model. The ω_1-Chang model ω_1-$\mathbb{C}$ is obtained by closing under ω_1-sequences of ordinals.

Question 5.6. What can be said about the ω_1-Chang model?

The question is due to Woodin (personal communication), as is most of the known information. Gitik has pointed out that (contrary to my earlier belief) his technique of recovering extenders from threads, or strings of indiscernibles, appears to be essentially unlimited for strings whose length has uncountable cofinality. It follows that the lower bound, the counterpart to Theorem 1.5(1), is probably at least as large as any cardinal for which there is a pure extender model.

There is one minor caveat to this statement:

Proposition 5.7. *Suppose that $V = L[\mathcal{E}]$ is an extender model, and that there is an iterated ultrapower $i\colon V \to M$ where M is a definable submodel of ω_1-$\mathbb{C}$. Then there is no strong cardinal in V.*

Proof. Suppose the contrary, and let κ be the smallest strong cardinal. Then $i(\kappa)$ is the smallest strong cardinal in M. However, since κ is strong there is an extender E with critical point κ such that $i^E(\kappa) > i(\kappa)$ and $^{\omega_1}\mathrm{Ult}(L[\mathcal{E}], E) \subseteq \mathrm{Ult}(L[\mathcal{E}], E)$. Then $\omega_1\text{-}\mathbb{C} = (\omega_1\text{-}\mathbb{C})^{\mathrm{Ult}(L[\mathcal{E}], E)}$, but the smallest strong cardinal in the latter is $i^E(i(\kappa)) \geqslant i^E(\kappa) > i(\kappa)$. □

However this observation has no implications for the existence of a sharp for $\omega_1\text{-}\mathbb{C}$. For example, if $V = L[\mathcal{E}]$ where $\mathcal{E}$ is a proper set, then so long as $K^{\omega_1\text{-}\mathbb{C}}$ exists and is sufficiently iterable, Gitik's technique gives an iterated ultrapower from $L[\mathcal{E}]$ to $K^{\omega_1\text{-}\mathbb{C}}$.

Woodin has observed that the existence of a sharp for $\omega_1\text{-}\mathbb{C}$ would imply the Axiom of Determinacy, which implies that there is no embedding from ω_1 into the reals in $\omega_1\text{-}\mathbb{C}$, and hence none in V. Thus a sharp for $\omega_1\text{-}\mathbb{C}$ is inconsistent with the Axiom of Choice in V. However it would be of interest to find a sharp for the ω_1-Chang model as defined inside an inner model which satisfies the Axiom of Determinacy.

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